SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.19.01.05
Vol.19 (32), No.1 (2023), 63–77

FAST GROWTH OF THE LOGARITHMIC DERIVATIVE WITH APPLICATIONS TO COMPLEX DIFFERENTIAL EQUATIONS

MOHAMED ABDELHAK KARA AND BENHARRAT BELAÏDI

ABSTRACT. In this paper, we give some estimates on the growth of the logarithmic derivative of meromorphic functions by considering the concept of φ-order. We discuss their relationship with the growth of solutions of certain complex differential equations.

1. INTRODUCTION AND MAIN RESULTS

Throughout this paper, we assume familiarities of the reader with the fundamental results and standard notations of Nevanlinna value distribution theory such as $M(r,f), T(r,f), n(r,a,f), N(r,f), \overline{N}(r,f)$ (see [10, 18]). Also, the term "meromorphic function" will mean "meromorphic function in the whole complex plane $\mathbb{C}$". We recall the following definitions.

Definition 1.1. [18] *The order of a meromorphic function f is defined by*

$$\rho(f) := \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log r}.$$

If f is an entire function, then the order of f is given by

$$\widetilde{\rho}(f) := \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log r} = \rho(f).$$

In addition, we define the logarithmic measure of a set $E \subset (1,+\infty)$ by $mes_l(E) = \int_E \frac{dt}{t}$ and the linear measure of a set $F \subset (0,+\infty)$ by $mes(F) = \int_F dt$. The following result due to Gundersen [8] plays an important role in the theory of complex differential equations.

Theorem 1.1. [8] *Let f be a transcendental meromorphic function of finite order $\rho := \rho(f)$. Let $\varepsilon > 0$ be a constant, and k, j be integers such that $k > j \geq 0$. Then, the following hold:*

2010 *Mathematics Subject Classification.* 34M10, 30D35.

Key words and phrases. meromorphic function, φ-order, φ-convergence exponent, logarithmic derivative, complex differential equation.

(i) *There exists a set* $E \subset (1,+\infty)$ *that has finite logarithmic measure, such that for all* z *satisfying* $|z| \notin E \cup [0,1]$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}.$$

(ii) *There exists a set* $F \subset (0,+\infty)$ *that has finite linear measure, such that for all* z *satisfying* $|z| \notin F$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho+\varepsilon)}.$$

In [12], Latreuch and Belaïdi obtained certain sharpness estimates for the growth of the logarithmic derivative of meromorphic functions.

Theorem 1.2. [12] *Let* f *be a meromorphic function and* k *be an integer. Then*

$$\max\left\{\rho\left(\frac{f^{(k)}}{f}\right), k \geq 2\right\} = \rho\left(\frac{f'}{f}\right).$$

Very recently, some authors [3, 4, 9, 14, 20] have used a more general concept to measure the growth of complex functions called the φ-order (cf. [19]). They employed this new concept in the investigation of fast growing solutions of higher order complex linear differential equations.

Definition 1.2. [9] *Let* φ *be an increasing unbounded function on* $(0,+\infty)$. *The* φ*-orders of a meromorphic function* f *are defined by*

$$\rho_{\varphi}^{0}(f) := \limsup_{r\to+\infty}\frac{\varphi\left(e^{T(r,f)}\right)}{\log r}, \qquad \rho_{\varphi}^{1}(f) := \limsup_{r\to+\infty}\frac{\varphi(T(r,f))}{\log r}.$$

If f *is an entire function, then the* φ*-orders are defined by*

$$\tilde{\rho}_{\varphi}^{0}(f) := \limsup_{r\to+\infty}\frac{\varphi(M(r,f))}{\log r}, \qquad \tilde{\rho}_{\varphi}^{1}(f) := \limsup_{r\to+\infty}\frac{\varphi(\log M(r,f))}{\log r}.$$

Example 1.1. *For all* $r \in (0,+\infty)$ *large enough, we define* $\log_0 r = r$ *and* $\log_p r = \log(\log_{p-1} r)$, *where* $p \in \mathbb{N}$. *If we choose* $\varphi(r) = \log_p r$ $(p \geq 2)$, *then* $\rho_{\varphi}^{1}(f) = \rho_{\log_p}^{1}(f) := \rho_p(f)$ *which is well known as the iterated* p*-order of* f [5, 15]. *In particular,* $\rho_{\log_2}^{0}(f) = \rho_1(f) = \rho(f)$ *is the usual order of* f *and* $\rho_{\log_2}^{1}(f) = \rho_2(f)$ *is the hyper-order of* f [6, 7].

We use the symbol Φ to denote the class of positive unbounded increasing functions on $(0,+\infty)$, such that $\varphi(e^x)$ grows slowly, *i.e.*,

$$\forall c > 0, \lim_{x\to+\infty}\frac{\varphi(e^{cx})}{\varphi(e^x)} = 1.$$

For instance, $\varphi(r) = \log r \notin \Phi$ and $\varphi(r) = \log_p r$, $(p \geq 2)$ belongs to the class Φ. In [1], Bandura et al. proved that for any entire transcendental function f of infinite order $\rho(f) = +\infty$, there exists $\varphi \in \Phi$ satisfying $\rho_\varphi^0(f) < +\infty$.

Proposition 1.1. [9,20] *If* $\varphi \in \Phi$, *then*

$$\forall m > 0, \forall k \geq 0 : \frac{\varphi^{-1}(\log x^m)}{x^k} \longrightarrow +\infty,\ x \longrightarrow +\infty. \tag{1.1}$$

$$\forall \delta > 0 : \frac{\log \varphi^{-1}((1+\delta)x)}{\log \varphi^{-1}(x)} \longrightarrow +\infty,\ x \longrightarrow +\infty. \tag{1.2}$$

$$\forall \delta > 0,\ \varphi(\delta x) \leq \varphi(x^\delta) \leq (1+o(1))\varphi(x),\ x \longrightarrow +\infty. \tag{1.3}$$

$$\varphi\left(e^x\right) = O(x),\ x \longrightarrow +\infty. \tag{1.4}$$

Proposition 1.2. [9] *Let* $\varphi \in \Phi$ *and* f *be an entire function. Then*

$$\rho_\varphi^j(f) = \tilde{\rho}_\varphi^j(f),\ j = 0,1.$$

Definition 1.3. [14] *Let* φ *be an increasing unbounded function on* $(0,+\infty)$. *The* φ*-convergence exponents of the sequence of zeros of a meromorphic function* f *are defined by*

$$\lambda_\varphi^0(f) := \limsup_{r\to+\infty} \frac{\varphi\left(e^{N\left(r,\frac{1}{f}\right)}\right)}{\log r}, \qquad \lambda_\varphi^1(f) := \limsup_{r\to+\infty} \frac{\varphi\left(N\left(r,\frac{1}{f}\right)\right)}{\log r}.$$

Similarly, the φ*-convergence exponents of the sequence of distinct zeros of* f *are defined by*

$$\overline{\lambda}_\varphi^0(f) := \limsup_{r\to+\infty} \frac{\varphi\left(e^{\overline{N}\left(r,\frac{1}{f}\right)}\right)}{\log r}, \qquad \overline{\lambda}_\varphi^1(f) := \limsup_{r\to+\infty} \frac{\varphi\left(\overline{N}\left(r,\frac{1}{f}\right)\right)}{\log r}.$$

In [14], the authors investigated the fast growth and the oscillation of solutions of the non-homogeneous differential equation

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = F(z) \tag{1.5}$$

and obtained the following theorem.

Theorem 1.3. [14] *Let* $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ *be meromorphic functions and let* f *be a meromorphic solution of equation* (1.5). *If*

$$\max\left\{\rho_\varphi^1(F), \rho_\varphi^1(A_j) : j = 0,1,\ldots,k-1\right\} < \rho_\varphi^1(f),$$

then we have

$$\overline{\lambda}_\varphi^1(f) = \lambda_\varphi^1(f) = \rho_\varphi^1(f).$$

By using the same arguments of the proof of Theorem 1.3 ([14], Lemma 8), we obtain the following result.

Theorem 1.4. *Under the assumptions of Theorem 1.3, if*

$$\max\left\{\rho_\varphi^0(F), \rho_\varphi^0(A_j) : j = 0, 1, \ldots, k-1\right\} < \rho_\varphi^0(f),$$

then we have

$$\overline{\lambda}_\varphi^0(f) = \lambda_\varphi^0(f) = \rho_\varphi^0(f).$$

In this paper, we make use of the concepts of φ-order and φ-convergence exponents to establish some estimates on the fast growth and the oscillation of logarithmic derivative of meromorphic functions. As applications, we describe the connection between these estimates and the solutions of some complex differential equations.

Theorem 1.5. *Let $\varphi \in \Phi$ and f be a meromorphic function. For any integer $k \geq 2$, we have*

$$\begin{aligned} \rho_\varphi^j\left(\frac{f'}{f}\right) &= \max\left\{\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right), \rho_\varphi^j\left(\frac{f^{(k+1)}}{f}\right)\right\} \\ &= \max\left\{\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right)\right\}, (j = 0, 1). \end{aligned}$$

Theorem 1.6. *Let $\varphi \in \Phi$ and f be a meromorphic function. If there exists an integer $k \geq 1$ such that $\rho_\varphi^0\left(\frac{f^{(k)}}{f}\right) = \rho_\varphi^0(f) > \rho_\varphi^1(f)$, then we have*

$$\max\left\{\overline{\lambda}_\varphi^0(f), \overline{\lambda}_\varphi^0\left(\frac{1}{f}\right)\right\} = \max\left\{\lambda_\varphi^0(f), \lambda_\varphi^0\left(\frac{1}{f}\right)\right\} = \rho_\varphi^0(f). \tag{1.6}$$

Moreover, if f is an entire function, then $\overline{\lambda}_\varphi^0(f) = \lambda_\varphi^0(f) = \rho_\varphi^0(f)$.

Theorem 1.7. *Let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be entire functions and let $\varphi \in \Phi$. If f is a solution of equation* (1.5) *satisfying*

$$\max\left\{\rho_\varphi^0(F), \rho_\varphi^0(A_j) : j = 0, 1, \ldots, k-1\right\} < \rho_\varphi^0(f),$$

then

$$\rho_\varphi^0\left(\frac{f'}{f}\right) = \rho_\varphi^0(f) = \overline{\lambda}_\varphi^0(f) = \lambda_\varphi^0(f).$$

Moreover, if $\dfrac{f^{(i)}}{f}$ is not constant for any integer $i \geq 2$, then

$$\rho_\varphi^0\left(\frac{f^{(i)}}{f}\right) = \rho_\varphi^0(f) = \overline{\lambda}_\varphi^0(f) = \lambda_\varphi^0(f). \tag{1.7}$$

Theorem 1.8. *Let $n \geq 2$ be an integer and let A_j $(j = 1, \ldots, n)$ be meromorphic functions. If f is a non-zero meromorphic solution of the differential equation ($k \geq 1$ is an integer)*

$$f^{(k)} = A_1 f + A_2 f^2 + \cdots + A_{n-1} f^{n-1} + A_n f^n \tag{1.8}$$

satisfying

$$\max\left\{\rho_{\varphi}^{0}(A_j) : j=1,\ldots,n\right\} < \rho_{\varphi}^{0}(f) < +\infty, \tag{1.9}$$

then for $k \geq 1$, *we have*

$$\rho_{\varphi}^{0}\left(\frac{f^{(k)}}{f}\right) = \rho_{\varphi}^{0}(f) = \max\left\{\overline{\lambda}_{\varphi}^{0}(f), \overline{\lambda}_{\varphi}^{0}\left(\frac{1}{f}\right)\right\} = \max\left\{\lambda_{\varphi}^{0}(f), \lambda_{\varphi}^{0}\left(\frac{1}{f}\right)\right\}. \tag{1.10}$$

Remark 1.1. For $\varphi(r) = \log\log r$, it is clear that Theorem 1.5 extends and generalizes Theorem 1.2 from the usual order to the concept of φ-order. However, Theorem 1.7 improves Theorem 1.4.

2. Basic lemmas

Lemma 2.1. [2, 18] *Let* $g : (0,+\infty) \to \mathbb{R}$ *and* $h : (0,+\infty) \to \mathbb{R}$ *be monotone non-decreasing functions such that* $g(r) \leq h(r)$ *outside of an exceptional set* $F_1 \subset (0,+\infty)$ *of finite linear measure. Then, for any* $\alpha > 1$, *there exists* $r_0 > 0$ *such that* $g(r) \leq h(\alpha r)$ *for all* $r > r_0$.

Lemma 2.2. [16] *Let* f *and* $a_0,\ldots,a_n$ *be meromorphic functions, where* $n \geq 1$ *is an integer. If* $a_n(z) \not\equiv 0$ *and* $F = a_0 + a_1 f + \cdots + a_n f^n$, *then*

$$T(r,F) = nT(r,f) + O\left(\sum_{k=0}^{n} T(r,a_k)\right).$$

Lemma 2.3. [9,13] *Let* $\varphi \in \Phi$ *and* f_1, f_2 *be two meromorphic functions. Then, for* $j = 0,1$ *the following statements hold:*

(i) $\rho_{\varphi}^{j}\left(\frac{1}{f_1}\right) = \rho_{\varphi}^{j}(f_1)$, $f_1 \not\equiv 0$,

(ii) $\rho_{\varphi}^{j}(f_1') = \rho_{\varphi}^{j}(f_1)$,

(iii) $\max\left\{\rho_{\varphi}^{j}(f_1+f_2), \rho_{\varphi}^{j}(f_1 f_2)\right\} \leq \max\{\rho_{\varphi}^{j}(f_1), \rho_{\varphi}^{j}(f_2)\}$,

(iv) *if* $\rho_{\varphi}^{j}(f_1) < \rho_{\varphi}^{j}(f_2)$, *then* $\rho_{\varphi}^{j}(f_1+f_2) = \rho_{\varphi}^{j}(f_1 f_2) = \rho_{\varphi}^{j}(f_2)$.

Lemma 2.4. [14] *Let* f *be a meromorphic function. If* $\rho_{\varphi}^{0}(f) < +\infty$, *then* $\rho_{\varphi}^{1}(f) = 0$.

Lemma 2.5. [9] *Let* $\varphi \in \Phi$ *and* f *be a meromorphic function of order* $\rho := \rho_{\varphi}^{1}(f)$. *Then, for any given* $\varepsilon > 0$ *and for any integer* $k \geq 1$, *we have that*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log \varphi^{-1}\left(\log r^{\rho+\varepsilon}\right)\right)$$

holds possibly outside of an exceptional set $F_2 \subset (0,+\infty)$ *of finite linear measure.*

Lemma 2.6. [17] *Let* $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ *be an entire function. Let* $\mu(r)$ *and* $\nu_f(r)$ *denote respectively the maximum term and the central index of* f, *i.e.,* $\mu(r) = \max\{|a_n| r^n; n = 0, 1, \dots\}$ *and* $\nu_f(r) = \max\{n : \mu(r) = |a_n| r^n\}$. *Then, we have*

$$\log \mu(r) = \log|a_0| + \int_0^r \frac{\nu_f(t)}{t}\, dt \quad (|a_0| \neq 0), \tag{2.1}$$

$$M(r,f) < \mu(r)\left\{\nu_f(R) + \frac{R}{R-r}\right\} \quad (R > r). \tag{2.2}$$

Lemma 2.7. [11, 18] *Let* f *be a transcendental entire function and let* z *be a point with* $|z| = r$ *at which* $|f(z)| = M(r,f)$. *Then*

$$\frac{f^{(k)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^k (1 + o(1)) \quad (k \in \mathbb{N}) \tag{2.3}$$

holds for all $|z| = r$ *outside a set* $E_1 \subset (1, +\infty)$ *of* r *of finite logarithmic measure.*

Lemma 2.8. *Let* $\varphi \in \Phi$ *and* f *be an entire function such that* $\rho_\varphi^0(f) = +\infty$ *and* $\rho_\varphi^1(f) < +\infty$. *Then, we have*

$$\rho_\varphi^1(f) = \limsup_{r\to+\infty} \frac{\varphi(\nu_f(r))}{\log r},$$

where $\nu_f(r)$ *is the central index of* f.

Proof. Denote $\rho := \limsup_{r\to+\infty} \frac{\varphi(\nu_f(r))}{\log r}$. Then, for any given $\varepsilon > 0$ and sufficiently large r, we have

$$\nu_f(r) \le \varphi^{-1}(\log r^{\rho+\varepsilon}). \tag{2.4}$$

By setting $R = 2r$ in (2.2), we get

$$M(r,f) < \mu(r)\left(\nu_f(2r) + 2\right) = |a_{\nu_f(r)}| r^{\nu_f(r)} \left(\nu_f(2r) + 2\right). \tag{2.5}$$

Since $\{|a_n|\}_{n\ge 0}$ is a bounded sequence, then by using (1.1), (2.4) and (2.5), we obtain

$$\begin{aligned} \log M(r,f) &< \nu_f(r)\log r + \log \nu_f(2r) + c_1 \\ &\le \varphi^{-1}\left(\log r^{\rho+\varepsilon}\right)\log r + \log\left(\varphi^{-1}\left\{\log(2r)^{\rho+\varepsilon}\right\}\right) + c_1 \\ &\le \varphi^{-1}\left(\log r^{\rho+2\varepsilon}\right) + \varphi^{-1}\left(\log(2r)^{\rho+\varepsilon}\right) \\ &\le \varphi^{-1}\left(\log r^{\rho+3\varepsilon}\right), \end{aligned} \tag{2.6}$$

where $c_1 > 0$ is a real constant. From (2.6), by the monotonicity of φ, we get

$$\frac{\varphi(\log M(r,f))}{\log r} \le \rho + 3\varepsilon.$$

By the arbitrariness of $\varepsilon > 0$ and Proposition 1.2, we obtain

$$\rho_{\varphi}^{1}(f) \leq \rho := \limsup_{r\to+\infty} \frac{\varphi(\nu_f(r))}{\log r}. \tag{2.7}$$

Now, we prove the reverse inequality. Without loss of generality, we may assume $|a_0| \neq 0$. It follows from (2.1) that

$$\log \mu(2r) = \log|a_0| + \int_0^{2r} \frac{\nu_f(t)}{t}\,dt \geq \log|a_0| + \nu_f(r)\int_r^{2r} \frac{dt}{t}$$
$$= \log|a_0| + \nu_f(r)\log 2.$$

By Cauchy's inequality we have $\mu(2r) \leq M(2r, f)$ and then

$$\nu_f(r) \leq \frac{\log M(2r,f)}{\log 2} - \frac{\log|a_0|}{\log 2} \leq c_2 \log M(2r,f), \tag{2.8}$$

where $c_2 > 2$ is a real constant. It follows from (2.8) and Proposition 1.1, especially (1.3), that

$$\frac{\varphi(\nu_f(r))}{\log r} \leq \frac{(1+o(1))\varphi(\log M(2r,f))}{\log 2r} \cdot \frac{\log 2r}{\log r}.$$

Hence

$$\limsup_{r\to+\infty} \frac{\varphi(\nu_f(r))}{\log r} \leq \limsup_{r\to+\infty} \frac{\varphi(\log M(2r,f))}{\log 2r} = \rho_{\varphi}^{1}(f). \tag{2.9}$$

We deduce from (2.7) and (2.9) that

$$\rho_{\varphi}^{1}(f) = \limsup_{r\to+\infty} \frac{\varphi(\nu_f(r))}{\log r}. \qquad \square$$

Lemma 2.9. *Let $\varphi \in \Phi$ and f be an entire function such that $\rho_{\varphi}^{0}(f) = +\infty$ and $\rho := \rho_{\varphi}^{1}(f) < +\infty$. Then, there exists a set $E_2 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $r \in E_2$, we have*

$$\lim_{\substack{r\to+\infty \\ r\in E_2}} \frac{\varphi(\nu_f(r))}{\log r} = \rho \tag{2.10}$$

and

$$\lim_{\substack{r\to+\infty \\ r\in E_2}} \frac{\varphi\left(e^{\nu_f(r)}\right)}{\log r} = +\infty. \tag{2.11}$$

Proof. Lemma 2.8 implies that there exists a sequence $\{r_n, r_n \longrightarrow +\infty\}$ satisfying

$$\left(1+\frac{1}{n}\right) r_n < r_n \quad \text{and} \quad \lim_{r_n\to+\infty} \frac{\varphi(\nu_f(r_n))}{\log r_n} = \rho.$$

Then, there exists an integer $n_1 \geq 1$ such that for any $n \geq n_1$ and for any $r \in [r_n, (1+\frac{1}{n})r_n]$, we have

$$\frac{\varphi(\nu_f(r_n))}{\log(1+\frac{1}{n})r_n} \leq \frac{\varphi(\nu_f(r))}{\log r} \leq \frac{\varphi\left(\nu_f((1+\frac{1}{n})r_n)\right)}{\log r_n},$$

so

$$\begin{aligned}\frac{\varphi(\nu_f(r_n))}{\log r_n} \cdot \frac{\log r_n}{\log(1+\frac{1}{n})r_n} &\leq \frac{\varphi(\nu_f(r))}{\log r} \\ &\leq \frac{\varphi\left(\nu_f((1+\frac{1}{n})r_n)\right)}{\log(1+\frac{1}{n})r_n} \cdot \frac{\log(1+\frac{1}{n})r_n}{\log r_n}. \qquad (2.12)\end{aligned}$$

We set $E_2 = \bigcup_{n=n_1}^{+\infty} \left[r_n, (1+\frac{1}{n})r_n\right]$. By (2.12), we get

$$\lim_{\substack{r\to+\infty \\ r\in E_2}} \frac{\varphi(\nu_f(r))}{\log r} = \lim_{n\to+\infty} \frac{\varphi(\nu_f(r_n))}{\log r_n} = \rho,$$

where the logarithmic measure of E_2 satisfies

$$mes_l(E_2) = \int_{E_2} \frac{dr}{r} = \sum_{n=n_1}^{+\infty} \int_{r_n}^{(1+\frac{1}{n})r_n} \frac{dt}{t} = \sum_{n=n_1}^{+\infty} \log(1+\frac{1}{n}) = +\infty.$$

Moreover, for any given $\varepsilon > 0$ and sufficiently large $r \in E_2$, we have

$$\varphi^{-1}\left(\log r^{\rho-\varepsilon}\right) \leq \nu_f(r) \leq \varphi^{-1}\left(\log r^{\rho+\varepsilon}\right). \qquad (2.13)$$

Hence, it follows from (1.1), (1.4) and the left-hand side of (2.13) that

$$\begin{aligned}\lim_{\substack{r\to+\infty \\ r\in E_2}} \frac{\varphi\left(e^{\nu_f(r)}\right)}{\log r} &= \lim_{\substack{r\to+\infty \\ r\in E_2}} \frac{O(\nu_f(r))}{\log r} \geq \lim_{\substack{r\to+\infty \\ r\in E_2}} \frac{O\left(\varphi^{-1}\left(\log r^{\rho-\varepsilon}\right)\right)}{\log r} \\ &= \lim_{\substack{r\to+\infty \\ r\in E_2}} \left(\frac{O\left(\varphi^{-1}\left(\log r^{\rho-\varepsilon}\right)\right)}{r} \cdot \frac{r}{\log r}\right) = +\infty\end{aligned}$$

and therefore, (2.11) is fulfilled. $\square$

By similar discussion as in the first part of the proof of Lemma 2.9, we can easily prove the following lemma.

Lemma 2.10. *Let $\varphi \in \Phi$ and f be a meromorphic function with $\rho_\varphi^0(f) < +\infty$. Then, there exists a set $E_3 \subset (1, +\infty)$ with infinite logarithmic measure such that for all $r \in E_3$, we have*

$$\rho_\varphi^0(f) = \lim_{\substack{r\to+\infty \\ r\in E_3}} \frac{\varphi\left(e^{T(r,f)}\right)}{\log r}. \qquad (2.14)$$

Lemma 2.11. *Let $\varphi \in \Phi$ and f be an entire function such that $\rho_{\varphi}^{0}(f) = +\infty$. Then, for all sufficiently large $r \in E_4 \subset (1,+\infty)$ and for any $\gamma > 0$ large enough, we have*

$$M(r,f) > \left[\varphi^{-1}\left(\log\left(d_1 r^{\gamma}\right)\right)\right]^{d_2}, \tag{2.15}$$

where E_4 is of infinite logarithmic measure and d_1, d_2 are two positive constants.

Proof. In view of (2.8) and (2.11), for any sufficiently large number $\gamma > 0$, we have

$$c_2 \log M(r,f) \geq \nu_f\left(\frac{r}{2}\right) \geq \log \varphi^{-1}\left(\log\left(\frac{r}{2}\right)^{\gamma}\right) \quad (r \in E_4,\ r \longrightarrow +\infty),$$

where $c_2 > 2$ and $E_4 \subset (1,+\infty)$ is of infinite logarithmic measure. Thus, (2.15) follows immediately. □

Lemma 2.12. *Let $\varphi \in \Phi$ and let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be entire functions such that*

$$\alpha := \max\left\{\rho_{\varphi}^{0}(F), \rho_{\varphi}^{0}(A_j) : j = 0,1,\ldots,k-1\right\} < +\infty.$$

Then, every solution f of (1.5) *satisfies $\rho_{\varphi}^{1}(f) \leq \alpha$.*

Proof. In view of Lemma 2.4, if $\rho_{\varphi}^{0}(f) < +\infty$ then $\rho_{\varphi}^{1}(f) = 0 \leq \alpha$. Suppose that $\rho_{\varphi}^{0}(f) = +\infty$. By Lemma 2.7, there exists a set $E_1 \subset (1,+\infty)$ with $mes_l(E_1) < +\infty$ such that for all z satisfying $|z| = r \notin E_1$ and $|f(z)| = M(r,f)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^{j}(1+o(1)) \quad (j = 1,\ldots,k). \tag{2.16}$$

Since $\alpha := \max\left\{\rho_{\varphi}^{0}(F), \rho_{\varphi}^{0}(A_j) : j = 0,1,\ldots,k-1\right\} < +\infty$, then by Proposition 1.2, for any given $\varepsilon > 0$ and sufficiently large r, we have

$$|F(z)| \leq \varphi^{-1}\left(\log r^{\alpha+\varepsilon}\right) \text{ and } |A_j(z)| \leq \varphi^{-1}\left(\log r^{\alpha+\varepsilon}\right) \ (j = 0,\ldots,k-1). \tag{2.17}$$

We can write equation (1.5) as

$$\frac{f^{(k)}}{f} = \frac{F}{f} - A_{k-1}\frac{f^{(k-1)}}{f} - \cdots - A_1\frac{f'}{f} - A_0. \tag{2.18}$$

Substituting (2.15)–(2.17) into (2.18) yield

$$\begin{aligned}[\nu_f(r)]^{k}|1+o(1)| &\leq r^{k}\frac{\varphi^{-1}\left(\log r^{\alpha+\varepsilon}\right)}{\left[\varphi^{-1}\left(\log\left(d_1 r^{\gamma}\right)\right)\right]^{d_2}} \\ &\quad + kr[\nu_f(r)]^{k-1}|1+o(1)|\varphi^{-1}\left(\log r^{\alpha+\varepsilon}\right)\end{aligned}$$

for all $r \in E_4 \setminus E_1$, where $E_4 \subset (1,+\infty)$ is of infinite logarithmic measure. Choosing $\gamma > 2\alpha+1$, by Proposition 1.1 and the monotonicity of φ, it follows that

$$\limsup_{\substack{r \to +\infty \\ r \in E_4 \setminus E_1}} \frac{\varphi(\nu_f(r))}{\log r} \leq \alpha + 2\varepsilon.$$

Hence, by Lemma 2.9 we conclude that $\rho_{\varphi}^{1}(f) \leq \alpha$. □

3. Proofs of the main results

Proof of Theorem 1.5

For any integer $k \geq 2$, we have

$$\frac{f^{(k)}}{f} = \left(\frac{f^{(k-1)}}{f}\right)' + \left(\frac{f'}{f}\right)\left(\frac{f^{(k-1)}}{f}\right).$$

By Lemma 2.3, we obtain for $k \geq 2$ and $j = 0,1$

$$\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right) \leq \max\left\{\rho_\varphi^j\left(\frac{f'}{f}\right), \rho_\varphi^j\left(\frac{f^{(k-1)}}{f}\right)\right\}. \tag{3.1}$$

Similarly, we can get

$$\begin{aligned}\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right) &\leq \max\left\{\rho_\varphi^j\left(\frac{f'}{f}\right), \rho_\varphi^j\left(\frac{f^{(k-2)}}{f}\right)\right\}\\ &\ \vdots\\ &\leq \max\left\{\rho_\varphi^j\left(\frac{f'}{f}\right), \rho_\varphi^j\left(\frac{f'}{f}\right)\right\} = \rho_\varphi^j\left(\frac{f'}{f}\right).\end{aligned} \tag{3.2}$$

Now, we treat separately the following three cases.

Case 1 For $j = 0,1$, suppose that $\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right) < \rho_\varphi^j\left(\frac{f^{(k+1)}}{f}\right)$. For any integer $k \geq 1$, we have

$$\frac{f^{(k+1)}}{f} - \left(\frac{f^{(k)}}{f}\right)' = \left(\frac{f'}{f}\right)\left(\frac{f^{(k)}}{f}\right). \tag{3.3}$$

It follows from (3.2), (3.3) and Lemma 2.3 that

$$\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right) < \rho_\varphi^j\left(\frac{f^{(k+1)}}{f}\right) \leq \rho_\varphi^j\left(\frac{f'}{f}\right) \quad \text{and} \quad \rho_\varphi^j\left(\frac{f^{(k+1)}}{f}\right) = \rho_\varphi^j\left(\frac{f'}{f}\right).$$

Case 2 For $j = 0,1$, suppose that $\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right) = \rho_\varphi^j\left(\frac{f^{(k+1)}}{f}\right)$. By (3.2) we have $\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right) \leq \rho_\varphi^j\left(\frac{f'}{f}\right)$. Assume that $\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right) < \rho_\varphi^j\left(\frac{f'}{f}\right)$. Then, by (3.3) and Lemma 2.3 we obtain the contradiction $\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right) = \rho_\varphi^j\left(\frac{f'}{f}\right)$. Hence, $\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right) = \rho_\varphi^j\left(\frac{f'}{f}\right)$.

Case 3 For $j = 0,1$, suppose that $\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right) > \rho_\varphi^j\left(\frac{f^{(k+1)}}{f}\right)$. Again, by (3.3) and Lemma 2.3, we obtain

$$\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right) = \rho_\varphi^j\left(\frac{f'}{f}\frac{f^{(k)}}{f}\right). \tag{3.4}$$

By (3.2) we have $\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right) \leq \rho_\varphi^j\left(\frac{f'}{f}\right)$. Assume that $\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right) < \rho_\varphi^j\left(\frac{f'}{f}\right)$. Then, by (3.4) and Lemma 2.3, we get $\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right) = \rho_\varphi^j\left(\frac{f'}{f}\right)$ which is a contradiction. Hence, $\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right) = \rho_\varphi^j\left(\frac{f'}{f}\right)$.

Finally, for $j = 0, 1$ we deduce that

$$\rho_\varphi^j\left(\frac{f'}{f}\right) = \max\left\{\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right), \rho_\varphi^j\left(\frac{f^{(k+1)}}{f}\right)\right\}.$$

Moreover, by the last assertion, there exists some integer $n \geq 1$ satisfying $\rho_\varphi^j\left(\frac{f^{(n)}}{f}\right) = \rho_\varphi^j\left(\frac{f'}{f}\right)$ for $j = 0, 1$, and therefore,

$$\max\left\{\rho_\varphi^j\left(\frac{f^{(k)}}{f}\right),\ k \geq 2\right\} = \rho_\varphi^j\left(\frac{f'}{f}\right),$$

which completes the proof of Theorem 1.5. □

Proof of Theorem 1.6

Since there exists an integer $k \geq 1$ satisfying $\rho_\varphi^0(f) = \rho_\varphi^0\left(\frac{f^{(k)}}{f}\right)$, then by (3.2) we have $\rho_\varphi^0(f) \leq \rho_\varphi^0\left(\frac{f'}{f}\right)$. By Lemma 2.3 we obtain $\rho_\varphi^0\left(\frac{f'}{f}\right) \leq \rho_\varphi^0(f)$ and therefore

$$\rho_\varphi^0\left(\frac{f'}{f}\right) = \rho_\varphi^0(f). \tag{3.5}$$

On the other hand, it follows from Definition 1.3 and Lemma 2.5 that for any given $\varepsilon > 0$ and $r \notin F_2$, we have

$$\begin{aligned}
T\left(r, \frac{f'}{f}\right) &= m\left(r, \frac{f'}{f}\right) + N\left(r, \frac{f'}{f}\right) \\
&= m\left(r, \frac{f'}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) \\
&\leq O\left(\log \varphi^{-1}\left(\log r^{\rho+\varepsilon}\right)\right) + 2\log \varphi^{-1}\left(\log r^{\lambda+\varepsilon}\right) \\
&\leq O\left(\log \varphi^{-1}\left(\log r^{\max\{\rho,\lambda\}+3\varepsilon}\right)\right),
\end{aligned} \tag{3.6}$$

where $F_2 \subset (0, +\infty)$ is of finite linear measure, $\rho := \rho_\varphi^1(f)$ and $\lambda := \max\left\{\overline{\lambda}_\varphi^0(f), \overline{\lambda}_\varphi^0(\frac{1}{f})\right\}$. By the monotonicity of φ, Lemma 2.1, (1.3) and (3.6), we get that for any $\mu > 1$

$$\varphi\left(e^{T\left(r, \frac{f'}{f}\right)}\right) \leq (\max\{\rho, \lambda\} + 4\varepsilon)\log \mu r, \quad r \longrightarrow +\infty.$$

Hence, by the arbitrariness of $\varepsilon > 0$, we obtain

$$\begin{aligned}\rho_\varphi^0\left(\frac{f'}{f}\right) = \rho_\varphi^0(f) &\leq \max\left\{\rho_\varphi^1(f), \overline{\lambda}_\varphi^0(f), \overline{\lambda}_\varphi^0\left(\frac{1}{f}\right)\right\} \\ &\leq \max\left\{\rho_\varphi^1(f), \lambda_\varphi^0(f), \lambda_\varphi^0\left(\frac{1}{f}\right)\right\} \leq \rho_\varphi^0(f). \qquad (3.7)\end{aligned}$$

We finally deduce from (3.5) and (3.7) that

$$\rho_\varphi^0(f) = \max\left\{\overline{\lambda}_\varphi^0(f), \overline{\lambda}_\varphi^0\left(\frac{1}{f}\right)\right\} = \max\left\{\lambda_\varphi^0(f), \lambda_\varphi^0\left(\frac{1}{f}\right)\right\}.$$

If f is an entire function, it is obvious that $\overline{\lambda}_\varphi^0\left(\frac{1}{f}\right) = \lambda_\varphi^0\left(\frac{1}{f}\right) = 0$ and therefore

$$\rho_\varphi^0(f) = \overline{\lambda}_\varphi^0(f) = \lambda_\varphi^0(f). \qquad \square$$

Proof of Theorem 1.7

Equation (1.5) can be rewritten as

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0\right). \qquad (3.8)$$

It follows from Lemma 2.3 and (3.8) that

$$\rho_\varphi^0(f) \leq \max\left\{\rho_\varphi^0(F), \rho_\varphi^0(A_j), \rho_\varphi^0\left(\frac{f^{(i)}}{f}\right) : j = 0, 1, \ldots, k-1; i = 1, \ldots, k\right\}. \qquad (3.9)$$

Since $\max\left\{\rho_\varphi^0(F), \rho_\varphi^0(A_j) : j = 0, 1, \ldots, k-1\right\} < \rho_\varphi^0(f)$, then by (3.9), Theorem 1.5 and Lemma 2.3 we get

$$\rho_\varphi^0(f) \leq \max\left\{\rho_\varphi^0\left(\frac{f^{(i)}}{f}\right) : i = 1, \ldots, k\right\} = \rho_\varphi^0\left(\frac{f'}{f}\right) \leq \rho_\varphi^0(f). \qquad (3.10)$$

Thus,

$$\rho_\varphi^0\left(\frac{f'}{f}\right) = \rho_\varphi^0(f). \qquad (3.11)$$

We deduce from (3.11), Lemma 2.12 and Theorem 1.6 that

$$\rho_\varphi^0\left(\frac{f'}{f}\right) = \rho_\varphi^0(f) = \overline{\lambda}_\varphi^0(f) = \lambda_\varphi^0(f). \qquad (3.12)$$

On the other hand, we suppose that $\frac{f^{(i)}}{f} (i \geq 2)$ is not constant. Then, $\overline{n}(r, 0, f) \leq n(r, 0, \frac{f}{f^{(i)}})$ and therefore

$$\overline{N}\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{f^{(i)}}{f}\right) \leq T\left(r, \frac{f^{(i)}}{f}\right).$$

By the monotonicity of φ, we get for $i \geq 2$

$$\overline{\lambda}_{\varphi}^{0}(f) \leq \rho_{\varphi}^{0}\left(\frac{f^{(i)}}{f}\right). \tag{3.13}$$

It follows from (3.12), (3.13) and Lemma 2.3 that for $i \geq 2$

$$\rho_{\varphi}^{0}(f) = \overline{\lambda}_{\varphi}^{0}(f) = \lambda_{\varphi}^{0}(f) \leq \rho_{\varphi}^{0}\left(\frac{f^{(i)}}{f}\right) \leq \rho_{\varphi}^{0}(f).$$

Hence, (1.7) holds and Theorem 1.7 is proved. □

Proof of Theorem 1.8

Suppose that f is a non-zero meromorphic solution of equation (1.8). By the condition (1.9), we see that $\rho_{\varphi}^{0}(f) > 0$. It follows from (1.8) and Lemma 2.2 that

$$\begin{aligned} T\left(r, \frac{f^{(k)}}{f}\right) &= T\left(r, A_1 + A_2 f + \cdots + A_{n-1} f^{n-2} + A_n f^{n-1}\right) \\ &= (n-1)T(r,f) + O\left(\sum_{j=1}^{n} T(r,A_j)\right). \end{aligned} \tag{3.14}$$

Set $\alpha = \max\left\{\rho_{\varphi}^{0}(A_j) : j = 1, \ldots, n\right\}$. Then, for any given $\varepsilon > 0$ and sufficiently large r, we have

$$T(r,A_j) \leq \log \varphi^{-1}\left((\alpha+\varepsilon)\log r\right), \quad j = 1, \ldots, n. \tag{3.15}$$

By $\rho = \rho_{\varphi}^{0}(f)$ and Lemma 2.10, we obtain for sufficiently large $r \in E_3$ that

$$T(r,f) \geq \log \varphi^{-1}\left((\rho-\varepsilon)\log r\right), \tag{3.16}$$

where $E_3 \subset (1, +\infty)$ is of infinite logarithmic measure. For any ε $(0 < 2\varepsilon < \rho - \alpha)$, it follows from (3.15), (3.16) and (1.2) that

$$\frac{T(r,A_j)}{T(r,f)} \leq \frac{\log \varphi^{-1}\left((\alpha+\varepsilon)\log r\right)}{\log \varphi^{-1}\left((\rho-\varepsilon)\log r\right)} \longrightarrow 0, \quad (r \longrightarrow +\infty, r \in E_3, j = 1, \ldots, n). \tag{3.17}$$

For sufficiently large $r \in E_3$, we obtain from (3.14) and (3.17) that

$$T\left(r, \frac{f^{(k)}}{f}\right) = (n-1)T(r,f) + o\left(T(r,f)\right),$$

so

$$T(r,f) = \frac{1}{n-1+o(1)} T\left(r, \frac{f^{(k)}}{f}\right) = O\left(T\left(r, \frac{f^{(k)}}{f}\right)\right). \tag{3.18}$$

Hence, by the monotonicity of φ, (1.3) and (3.18), we get

$$\rho_{\varphi}^{0}(f) \leq \rho_{\varphi}^{0}\left(\frac{f^{(k)}}{f}\right). \tag{3.19}$$

On the other hand, Lemma 2.3 yields

$$\rho_{\varphi}^{0}\left(\frac{f^{(k)}}{f}\right) \leq \max\left\{\rho_{\varphi}^{0}\left(f^{(k)}\right), \rho_{\varphi}^{0}\left(\frac{1}{f}\right)\right\} = \rho_{\varphi}^{0}(f). \tag{3.20}$$

Hence, by (3.19) and (3.20), we deduce that $\rho_{\varphi}^{0}\left(\frac{f^{(k)}}{f}\right) = \rho_{\varphi}^{0}(f)$ for $k \geq 1$. Since $0 < \rho_{\varphi}^{0}(f) < +\infty$, by Lemma 2.4, we have $\rho_{\varphi}^{1}(f) = 0 < \rho_{\varphi}^{0}(f)$. Furthermore, one can see that (1.10) follows immediately from Theorem 1.6. □

Acknowledgement. The authors want to thank the editor and the anonymous referees for their constructive comments and suggestions, which greatly improved this article. This paper was supported by the Directorate-General for Scientific Research and Technological Development(DGRSDT).

REFERENCES

[1] A. Bandura, O. Skaskiv and P. Filevych, *Properties of entire solutions of some linear PDE's*, J. Appl. Math. Comput. Mech., 16 (2) (2017), 17–28.

[2] S. Bank, *A general theorem concerning the growth of solutions of first-order algebraic differential equations*, Compositio Math., 25 (1972), 61–70.

[3] B. Belaïdi, *Growth of ρ_{φ}-order solutions of linear differential equations with entire coefficients*, PanAmer. Math. J., 27 (4) (2017), 26–42.

[4] B. Belaïdi, *Fast growing solutions to linear differential equations with entire coefficients having the same φ-order*, J. Math. Appl., 42 (2019), 63–77.

[5] L. G. Bernal, *On growth k-order of solutions of a complex homogeneous linear differential equation*, Proc. Amer. Math. Soc., 101 (2) (1987), 317–322.

[6] C. H. Li and Y. X. Gu, *On the complex oscillation of differential equations $f'' + e^{az}f' + Q(z)f = F(z)$*, Acta Math. Sci., 25A (2) (2005), 192–200. (in Chinese)

[7] Z. X. Chen and C. C. Yang, *Some further results on the zeros and growths of entire solutions of second order linear differential equations*, Kodai Math. J., 22 (2) (1999), 273–285.

[8] G. G. Gundersen, *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates*, J. London Math. Soc., 37 (2) (1988), 88–104.

[9] I. Chyzhykov and N. Semochko, *Fast growing entire solutions of linear differential equations*, Math. Bull. Shevchenko Sci. Soc., 13 (2016), 1–16.

[10] W. K. Hayman, *Meromorphic Functions*, Oxford: Oxford Mathematical Monographs, Clarendon Press, 1964.

[11] W. K. Hayman, *The local growth of power series: a survey of the Wiman-Valiron method*, Canad. Math. Bull., 17 (3) (1974), 317–358.

[12] Z. Latreuch and B. Belaïdi, *Growth of logarithmic derivative of meromorphic functions*, Math. Scand., 113 (2) (2013), 248–261.

[13] M. A. Kara and B. Belaïdi, *Some estimates of the φ-order and the φ-type of entire and meromorphic functions*, Int. J. Open Problems Complex Analysis, 10 (3) (2019), 42–58.

[14] M. A. Kara and B. Belaïdi, *Growth of φ-order solutions of linear differential equations with meromorphic coefficients on the complex plane*, Ural Math. J., 6 (1) (2020), 95–113.

[15] L. Kinnunen, *Linear differential equations with solutions of finite iterated order*, Southeast Asian Bull. Math., 22 (4) (1998), 385–405.

[16] A. Z. Mokhon'ko and V. D. Mokhon'ko, *Estimates for the Nevanlinna characteristics of some classes of meromorphic functions and their applications to differential equations*, Sib. Math J., 15 (1974), 921–934.

[17] G. Jank and L. Volkmann, *Einführung in die Theorie der ganzen und Meromorphen Funktionen mit Anwendungen auf Differentialgleichungen*, Birkhäuser Verlag, Basel, 1985.

[18] I. Laine, *Nevanlinna theory and complex differential equations*, De Gruyter Studies in Mathematics 15, Walter de Gruyter & Co., Berlin, 1993.

[19] M. N. Sheremeta, *Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion*, Izv. Vysš. Učebn. Zaved. Matematika, 57 (2) (1967), 100–108. (in Russian).

[20] N. Semochko, *On the solution of linear differential equations of arbitrary fast growth in the unit disc*, Mat. Stud., 45 (1) (2016), 3–11.

(Received: July 20, 2021)
(Revised: December 20, 2021)

Mohamed Abdelhak Kara
Department of Mathematics
Laboratory of Pure and Applied Mathematics
University of Mostaganem (UMAB)
B. P. 227 Mostaganem, Algeria.
e-mail: *mohamed.kara.etu@univ-mosta.dz*
and
Benharrat Belaïdi
Department of Mathematics
Laboratory of Pure and Applied Mathematics
University of Mostaganem (UMAB)
B. P. 227 Mostaganem, Algeria.
e-mail: *benharrat.belaidi@univ-mosta.dz*