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GENERALIZED RELATIVE ORDER (α,β) ORIENTED GROWTH ANALYSIS OF COMPOSITE ENTIRE FUNCTIONS

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ABSTRACT. The main aim of this paper is to study some growth properties of entire functions on the basis of their maximum modulus and generalized relative order (α,β).

1. INTRODUCTION, DEFINITIONS AND NOTATIONS

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function $M_f(r)$ of $f = \sum\limits_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined as $M_f = \max\limits_{|z|=r}|f(z)|$. Moreover, if f is non-constant entire, then $M_f(r)$ is also strictly an increasing and continuous function of r. Therefore its inverse $M_f^{-1} : (M_f(0),\infty) \to (0,\infty)$ exists and is such that $\lim\limits_{s\to+\infty} M_f^{-1}(s) = \infty$. We use the standard notations and definitions from the theory of entire functions which are available in [11] and [12], and therefore we do not explain those in details.

Let L be a class of continuous non-negative on $(-\infty,+\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0,+\infty)$, i.e., α is a slowly increasing function. Clearly $L^0 \subset L$. Moreover we assume that throughout the present paper α and β always denote the functions belonging to L^0 unless otherwise specifically stated. The value

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} (\alpha \in L,\ \beta \in L)$$

is called [10] the generalized order (α,β) of f. Several authors made close investigations on the properties of entire functions related to the generalized order (α,β) in some different direction.

For the purpose of further applications of the generalized order (α,β) of an entire function, Biswas et al. [4] rewrite the definition of generalized order (α,β) of an entire function after giving a minor modification to the original definition of the generalized order (α,β) of an entire function (e.g. see, [10]).

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Definition 1.1. *[4] The generalized order* (α,β) *and generalized lower order* (α,β) *of an entire function* f *are defined as:*

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty}\frac{\alpha(M_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty}\frac{\alpha(M_f(r))}{\beta(r)}.$$

Mainly the growth investigation of entire functions has usually been done through their maximum moduli function in comparison with those of exponential functions. But if one is paying attention to evaluate the growth rates of any entire function with respect to a new entire function, the notions of relative growth indicators (see e.g. [1,2]) will come. Now in order to make some progress in the study of relative order, one may introduce the definitions of generalized relative order (α,β) and generalized relative lower order (α,β) of an entire function with respect to another entire function in the following way:

Definition 1.2. *[5] The generalized relative order* (α,β) *and generalized relative lower order* (α,β) *of an entire function* f *with respect to an entire function* g *are defined as:*

$$\rho_{(\alpha,\beta)}[f]_g = \limsup_{r\to+\infty}\frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f]_g = \liminf_{r\to+\infty}\frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)}.$$

The main aim of this paper is to establish some newly developed results related to the growth rates of the composition of two entire functions on the basis of generalized relative order (α,β) and generalized relative lower order (α,β) of entire function with respect to another entire function which extend some earlier results (see, e.g., [3]). If fact some works on generalized relative order (α,β) related to the growth of entire Dirichlet series have been explored by Mulyava et al. (see, e.g., [7], [8]).

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. *[9] If* f *and* g *are any two entire functions with* $g(0)=0$. *Then*

$$M_{f(g)}(r) \geq M_g(\frac{r}{2}) \text{ for all sufficiently large values of } r.$$

Lemma 2.2. *[6] Let* f *and* g *be any two entire functions with* $g(0)=0$. *Also let* B *satisfy* $0<B<1$ *and* $c(B)=\frac{(1-B)^2}{4B}$. *Then for all sufficiently large values of* r,

$$M_f(c(B)M_g(Br)) \leq M_{f(g)}(r) \leq M_f(M_g(r)).$$

In addition if $B=\frac{1}{2}$, *then for all sufficiently large values of* r,

$$M_{f(g)}(r) \geq M_f(\frac{1}{8}M_g(\frac{r}{2})).$$

Lemma 2.3. *[2] Suppose f is an entire function and $A > 1$, $0 < B < A$. Then for all sufficiently large r,*

$$M_f(Ar) \geq BM_f(r).$$

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let f, g and h be any three entire functions such that*

$$\limsup_{r\to+\infty} \frac{\alpha(M_h^{-1}(M_g(r)))}{(\beta(r))^{\gamma}} = A, \text{ a real number } > 0 \tag{3.1}$$

and

$$\liminf_{r\to+\infty} \frac{\alpha(M_h^{-1}(M_f(r)))}{(\alpha(M_h^{-1}(r)))^{\eta+1}} = B, \text{ a real number } > 0 \tag{3.2}$$

for any γ, η satisfying $0 < \gamma < 1$, $\eta > 0$ and $\gamma(\eta+1) > 1$. Then

$$\rho_{(\alpha,\beta)}[f(g)]_h = +\infty.$$

Proof. From (3.1) we have for a sequence of values of r tending to infinity

$$\alpha(M_h^{-1}(M_g(r))) \geq (A-\varepsilon)(\beta(r))^{\gamma} \tag{3.3}$$

and from (3.2) we obtain for all sufficiently large values of r that

$$\alpha(M_h^{-1}(M_f(r))) \geq (B-\varepsilon)(\alpha(M_h^{-1}(r)))^{\eta+1}.$$

Since $M_g(r)$ is a continuous, increasing and unbounded function of r, we get from above for all sufficiently large values of r that

$$\alpha(M_h^{-1}(M_f(M_g(r)))) \geq (B-\varepsilon)(\alpha(M_h^{-1}(M_g(r))))^{\eta+1}. \tag{3.4}$$

Also $M_h^{-1}(r)$ is an increasing function of r, so it follows from Lemma 2.2, Lemma 2.3, (3.3) and (3.4) for a sequence of values of r tending to infinity that

$$\alpha(M_h^{-1}(M_{f(g)}(18r))) \geq \alpha(M_h^{-1}(M_f(M_g(r)))) \tag{3.5}$$

$$\text{i.e., } \alpha(M_h^{-1}(M_{f(g)}(18r))) \geq (B-\varepsilon)(\alpha(M_h^{-1}(M_g(r))))^{\eta+1}$$

$$\text{i.e., } \alpha(M_h^{-1}(M_{f(g)}(18r))) \geq (B-\varepsilon)[(A-\varepsilon)(\beta(r))^{\gamma}]^{\eta+1}$$

$$\text{i.e., } \alpha(M_h^{-1}(M_{f(g)}(18r))) \geq (B-\varepsilon)(A-\varepsilon)^{\eta+1}(\beta(r))^{\gamma(\eta+1)}$$

$$\text{i.e., } \frac{\alpha(M_h^{-1}(M_{f(g)}(18r)))}{\beta(r)} \geq \frac{(B-\varepsilon)(A-\varepsilon)^{\eta+1}(\beta(r))^{\gamma(\eta+1)}}{\beta(r)}$$

$$\text{i.e., } \limsup_{r\to+\infty} \frac{\alpha(M_h^{-1}(M_{f(g)}(18r)))}{\beta(r)} \geq \liminf_{r\to+\infty} \frac{(B-\varepsilon)(A-\varepsilon)^{\eta+1}(\beta(r))^{\gamma(\eta+1)}}{\beta(r)}.$$

Since $\varepsilon(>0)$ is arbitrary and $\gamma(\eta+1) > 1$ it follows from above that

$$\rho_{(\alpha,\beta)}[f(g)]_h = +\infty,$$

which proves the theorem. □

In the line of Theorem 3.1 one may state the following two theorems without proof :

Theorem 3.2. *Let f, g and h be any three entire functions such that*

$$\liminf_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_g(r)))}{(\beta(r))^{\gamma}}=A,\ \text{a real number }>0$$

and

$$\limsup_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_f(r)))}{(\alpha(M_h^{-1}(r)))^{\eta+1}}=B,\ \text{a real number }>0,$$

for any γ, η satisfying $0<\gamma<1$, $\eta>0$, and $\gamma(\eta+1)>1$. Then

$$\rho_{(\alpha,\beta)}[f(g)]_h=+\infty.$$

Theorem 3.3. *Let f, g and h be any three entire functions such that*

$$\liminf_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_g(r)))}{(\beta(r))^{\gamma}}=A,\ \text{a real number }>0$$

and

$$\liminf_{r\to+\infty}\frac{\alpha M_h^{-1}(M_f(r))}{(\alpha(M_h^{-1}(r)))^{\eta+1}}=B,\ \text{a real number }>0,$$

for any γ, η with $0<\gamma<1$, $\eta>0$ and $\gamma(\eta+1)>1$. Then

$$\lambda_{(\alpha,\beta)}[f(g)]_h=+\infty.$$

Theorem 3.4. *Let f, g and h be any three entire functions such that*

$$\limsup_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_g(r)))}{(\beta(r))^{\gamma}}=A,\ \text{a real number }>0 \tag{3.6}$$

and

$$\liminf_{r\to+\infty}\frac{\log[\frac{\alpha(M_h^{-1}(M_f(r)))}{\alpha(M_h^{-1}(r))}]}{[\alpha(M_h^{-1}(r))]^{\eta}}=B,\ \text{a real number }>0 \tag{3.7}$$

for any γ, η satisfying $\gamma>1$, $0<\eta<1$ and $\eta\gamma>1$. Then

$$\rho_{(\alpha,\beta)}[f(g)]_h=+\infty.$$

Proof. From (3.6) for a sequence of values of r tending to infinity we get that

$$\alpha(M_h^{-1}(M_g(r)))\geq(A-\varepsilon)(\beta(r))^{\gamma} \tag{3.8}$$

and from (3.7) we obtain for all sufficiently large values of r that

$$\begin{aligned}\log[\frac{\alpha(M_h^{-1}(M_f(r)))}{\alpha(M_h^{-1}(r))}] &\geq (B-\varepsilon)[\alpha(M_h^{-1}(r))]^{\eta}\\ \text{i.e., }\frac{\alpha(M_h^{-1}(M_f(r)))}{\alpha(M_h^{-1}(r))} &\geq \exp[(B-\varepsilon)[\alpha(M_h^{-1}(r))]^{\eta}].\end{aligned}$$

Since $M_g(r)$ is a continuous, increasing and unbounded function of r, we get from above for all sufficiently large values of r that

$$\frac{\alpha(M_h^{-1}(M_f(M_g(r))))}{\alpha(M_h^{-1}(M_g(r)))} \geq \exp[(B-\varepsilon)[\alpha(M_h^{-1}(M_g(r)))]^{\eta}]. \tag{3.9}$$

Also $M_h^{-1}(r)$ is an increasing function of r, so it follows from (3.5), (3.8) and (3.9) for a sequence of values of r tending to infinity that

$$\frac{\alpha(M_h^{-1}(M_{f(g)}(18r)))}{\beta(r)} \geq \frac{\alpha(M_h^{-1}(M_f(M_g(r))))}{\beta(r)}$$

$$i.e.,\ \frac{\alpha(M_h^{-1}(M_{f(g)}(18r)))}{\beta(r)} \geq \frac{\alpha(M_h^{-1}(M_f(M_g(r))))}{\alpha(M_h^{-1}(M_g(r)))} \cdot \frac{\alpha(M_h^{-1}(M_g(r)))}{\beta(r)}$$

$$i.e.,\ \frac{\alpha(M_h^{-1}(M_{f(g)}(18r)))}{\beta(r)} \geq \exp[(B-\varepsilon)[\alpha(M_h^{-1}(M_g(r)))]^{\eta}] \cdot \frac{(A-\varepsilon)(\beta(r))^{\gamma}}{\beta(r)}$$

$$i.e.,\ \frac{\alpha(M_h^{-1}(M_{f(g)}(18r)))}{\beta(r)} \geq \exp[(B-\varepsilon)(A-\varepsilon)^{\eta}(\beta(r))^{\eta\gamma}] \cdot \frac{(A-\varepsilon)(\beta(r))^{\gamma}}{\beta(r)}$$

$$i.e.,\ \frac{\alpha(M_h^{-1}(M_{f(g)}(18r)))}{\beta(r)} \geq \exp[(B-\varepsilon)(A-\varepsilon)^{\eta}(\beta(r))^{\eta\gamma-1}\beta(r)] \cdot \frac{(A-\varepsilon)(\beta(r))^{\gamma}}{\beta(r)}$$

$$i.e.,\ \frac{\alpha(M_h^{-1}(M_{f(g)}(18r)))}{\beta(r)} \geq (\exp\beta(r))^{(B-\varepsilon)(A-\varepsilon)^{\eta}(\beta(r))^{\eta\gamma-1}} \cdot \frac{(A-\varepsilon)(\beta(r))^{\gamma}}{\beta(r)}$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_{f(g)}(18r)))}{\beta(r)} \geq \liminf_{r\to+\infty}\left((\exp\beta(r))^{(B-\varepsilon)(A-\varepsilon)^{\eta}(\beta(r))^{\eta\gamma-1}} \cdot \frac{(A-\varepsilon)(\beta(r))^{\gamma}}{\beta(r)}\right).$$

Since $\varepsilon(>0)$ is arbitrary and $\gamma > 1$, $\eta\gamma > 1$, the theorem follows from the above. □

In the line of Theorem 3.4 one may also state the following two theorems without proof :

Theorem 3.5. *Let f, g and h be any three entire functions such that*

$$\liminf_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_g(r)))}{(\beta(r))^{\gamma}} = A, \text{ a real number } > 0$$

$$\text{and} \quad \limsup_{r\to+\infty}\frac{\log[\frac{\alpha(M_h^{-1}(M_f(r)))}{\alpha(M_h^{-1}(r))}]}{[\alpha(M_h^{-1}(r))]^{\eta}} = B, \text{ a real number } > 0$$

for any γ, η *with* $\gamma > 1$, $0 < \eta < 1$ *and* $\eta\gamma > 1$. *Then*

$$\rho_{(\alpha,\beta)}[f(g)]_h = +\infty.$$

Theorem 3.6. *Let* f, g *and* h *be any three entire functions such that*

$$\liminf_{r\to+\infty} \frac{\alpha(M_h^{-1}(M_g(r)))}{(\log^{[q+1]} r)^{\gamma}} = A, \textit{ a real number } > 0$$

$$\textit{and} \quad \liminf_{r\to+\infty} \frac{\log\left[\frac{\alpha(M_h^{-1}(M_f(r)))}{\alpha(M_h^{-1}(r))}\right]}{[\alpha(M_h^{-1}(r))]^{\eta}} = B, \textit{ a real number } > 0$$

for any γ, η *satisfying* $\gamma > 1$, $0 < \eta < 1$ *and* $\eta\gamma > 1$. *Then*

$$\lambda_{(\alpha,\beta)}[f(g)]_h = +\infty.$$

Theorem 3.7. *Let* f, g *and* h *be any three entire functions such that* $0 < \lambda_{(\alpha,\beta)}[g]_h \leq \rho_{(\alpha,\beta)}[g]_h < +\infty$ *and*

$$\limsup_{r\to+\infty} \frac{\alpha(M_h^{-1}(M_f(r)))}{\alpha(M_h^{-1}(r))} = A, \textit{ a real number } < +\infty.$$

Then

$$\lambda_{(\alpha,\beta)}[f(g)]_h \leq A\lambda_{(\alpha,\beta)}[g]_h \leq \rho_{(\alpha,\beta)}[f(g)]_h \leq A\rho_{(\alpha,\beta)}[g]_h.$$

Proof. Since $M_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.2 for all sufficiently large values of r that

$$M_h^{-1}(M_{f(g)}(r)) \leq M_h^{-1}(M_f(M_g(r))). \tag{3.10}$$

Now from (3.5) we get for all sufficiently large values of r that

$$\frac{\alpha(M_h^{-1}(M_{f(g)}(18r)))}{\beta(r)} \geq \frac{\alpha(M_h^{-1}(M_f(M_g(r))))}{\beta(r)}$$

$$i.e., \ \frac{\alpha(M_h^{-1}(M_{f(g)}(18r)))}{\beta(r)} \geq \frac{\alpha(M_h^{-1}(M_f(M_g(r))))}{\alpha(M_h^{-1}(M_g(r)))} \cdot \frac{\alpha(M_h^{-1}(M_g(r)))}{\beta(r)}$$

$$i.e., \ \limsup_{r\to+\infty} \frac{\alpha(M_h^{-1}(M_{f(g)}(18r)))}{\beta(r)} \geq \limsup_{r\to+\infty} \frac{\alpha(M_h^{-1}(M_f(M_g(r))))}{\alpha(M_h^{-1}(M_g(r)))} \cdot \liminf_{r\to+\infty} \frac{\alpha(M_h^{-1}(M_g(r)))}{\beta(r)}$$

$$i.e., \ \rho_{(\alpha,\beta)}[f(g)]_h \geq A \cdot \lambda_{(\alpha,\beta)}[g]_h. \tag{3.11}$$

Similarly from (3.10) it follows for all sufficiently large values of r that

$$\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\beta(r)} \leq \frac{\alpha(M_h^{-1}(M_f(M_g(r))))}{\alpha(M_h^{-1}(M_g(r)))} \cdot \frac{\alpha(M_h^{-1}(M_g(r)))}{\beta(r)} \tag{3.12}$$

$$i.e.,\ \liminf_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\beta(r)}$$
$$\leq \limsup_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_f(M_g(r))))}{\alpha(M_h^{-1}(M_g(r)))}\cdot\liminf_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_g(r)))}{\beta(r)}$$
$$i.e.,\ \lambda_{(\alpha,\beta)}[f(g)]_h \leq A.\lambda_{(\alpha,\beta)}[g]_h. \tag{3.13}$$

Also from (3.12) we obtain for all sufficiently large values of r that

$$\limsup_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\beta(r)} \leq \limsup_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_f(M_g(r))))}{\alpha(M_h^{-1}(M_g(r)))}\cdot\limsup_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_g(r)))}{\beta(r)}$$
$$i.e.,\ \rho_{(\alpha,\beta)}[f(g)]_h \leq A.\rho_{(\alpha,\beta)}[g]_h. \tag{3.14}$$

Therefore the theorem follows from (3.11), (3.13) and (3.14) . □

Theorem 3.8. *Let f, g and h be any three entire functions such that* $0<\lambda_{(\alpha,\beta)}[g]_h\leq \rho_{(\alpha,\beta)}[g]_h<+\infty$ *and*

$$\liminf_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_f(r)))}{\alpha(M_h^{-1}(r))}=A,\ \textit{a real number} <+\infty.$$

Then

$$\lambda_{(\alpha,\beta)}[f(g)]_h \leq A\rho_{(\alpha,\beta)}[g]_h \leq \rho_{(\alpha,\beta)}[f(g)]_h.$$

The proof of Theorem 3.8 is omitted because it can be carried out in the line of Theorem 3.7.

Theorem 3.9. *Let f, g and h be any three entire functions with $g(0)=0$. Then*

(*i*) $\rho_{(\alpha,\beta)}[f(g)]_h=+\infty$ *if* $\rho_{(\alpha,\beta)}[g]_h=+\infty$ *or*
(*ii*) $\rho_{(\alpha,\beta)}[f(g)]_h=+\infty$ *if* $\rho_{(\alpha,\beta)}[f]_h>0$ *and* $\lambda_{(\varphi,\beta)}[g]>0$

where $\exp(\varphi(\beta^{-1}(r)))<r$.

Proof. **Case I.** Let $\rho_{(\alpha,\beta)}[g]_h=\infty$.
Since $M_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2.1, for all sufficiently large values of r that

$$M_{f(g)}(r)\geq M_g(\frac{r}{2}) \text{ for all sufficiently large values of } r.$$

$$\frac{\alpha(M_h^{-1}(M_{f(g)}(2r)))}{\beta(r)}\geq\frac{\alpha(M_h^{-1}(M_g(r)))}{\beta(r)}$$
$$i.e.,\ \limsup_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_{f(g)}(2r)))}{\beta(r)}\geq\limsup_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_g(r)))}{\beta(r)}$$
$$i.e.,\ \rho_{(\alpha,\beta)}[f(g)]_h\geq\rho_{(\alpha,\beta)}[g]_h$$
$$i.e.,\ \rho_{(\alpha,\beta)}[f(g)]_h=+\infty.$$

This proves the first part of the theorem.

Case II. Suppose $\rho_{(\alpha,\beta)}[f]_h > 0$ and $\lambda_{(\varphi,\beta)}[g] > 0$.
As $M_h^{-1}(r)$ is an increasing function of r, we get from Lemma 2.2, for a sequence of values of r tending to infinity that

$$\begin{aligned}
&\alpha(M_h^{-1}(M_{f(g)}(18r))) \geq \alpha(M_h^{-1}(M_f(M_g(r))))\\
i.e.,\ &\alpha(M_h^{-1}(M_{f(g)}(18r))) \geq (\rho_{(\alpha,\beta)}[f]_h - \varepsilon)\beta(M_g(r))\\
i.e.,\ &\alpha(M_h^{-1}(M_{f(g)}(18r))) \geq (\rho_{(\alpha,\beta)}[f]_h - \varepsilon)\exp(\varphi(M_g(r)))\\
i.e.,\ &\alpha(M_h^{-1}(M_{f(g)}(18r))) \geq (\rho_{(\alpha,\beta)}[f]_h - \varepsilon)(\exp\beta(r))^{(\lambda_{(\varphi,\beta)}[g]-\varepsilon)}\\
i.e.,\ &\frac{\alpha(M_h^{-1}(M_{f(g)}(18r)))}{\beta(r)} \geq \frac{(\rho_{(\alpha,\beta)}[f]_h - \varepsilon)(\exp\beta(r))^{(\lambda_{(\varphi,\beta)}[g]-\varepsilon)}}{\beta(r)}\\
i.e.,\ &\rho_{(\alpha,\beta)}[f(g)]_h = +\infty,
\end{aligned}$$

which is the second part of the theorem. □

Corollary 3.1. *Let f, g and h be any three entire functions such that $g(0) = 0$, $\rho_{(\varphi,\beta)}[g] > 0$ and $\lambda_{(\alpha,\beta)}[f]_h > 0$ where $\exp(\varphi(\beta^{-1}(r))) < r$. Then*

$$\rho_{(\alpha,\beta)}[f(g)]_h = +\infty.$$

The proof of Corollary 3.1 is omitted as it can be carried out in the line of Theorem 3.9.

In the line of Theorem 3.9 one can easily prove the following theorem :

Theorem 3.10. *Let f, g and h be any three entire functions with $g(0) = 0$. Then*
(*i*) $\lambda_{(\alpha,\beta)}[f(g)]_h = +\infty$ *if* $\lambda_{(\alpha,\beta)}[g]_h = +\infty$ *or*
(*ii*) $\lambda_{(\alpha,\beta)}[f(g)]_h = +\infty$ *if* $\lambda_{(\alpha,\beta)}[f]_h > 0$ *and* $\lambda_{(\varphi,\beta)}[g] > 0$
where $\exp(\varphi(\beta^{-1}(r))) < r$.

Theorem 3.11. *Let f, g and h be any three entire functions such that $g(0) = 0$, $\rho_{(\alpha,\beta)}[f]_h > 0$ and $\lambda_{(\varphi,\beta)}[g] > 0$ where $\exp(\varphi(\beta^{-1}(r))) < r$. Then*

$$\limsup_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_h^{-1}(M_f(r)))} = +\infty.$$

Proof. In view of Theorem 3.9, we obtain that

$$\limsup_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_h^{-1}(M_f(r)))} \geq \limsup_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\beta(r)} \cdot \liminf_{r\to+\infty}\frac{\beta(r)}{\alpha(M_h^{-1}(M_f(r)))}$$

$$i.e., \limsup_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_h^{-1}(M_f(r)))} \geq \rho_{(\alpha,\beta)}[f(g)]_h \cdot \frac{1}{\rho_{(\alpha,\beta)}[f]_h}$$

$$i.e., \limsup_{r\to+\infty}\frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_h^{-1}(M_f(r)))} = +\infty.$$

Thus the theorem follows. □

Theorem 3.12. *Let f, g and h be any three entire functions such that $g(0) = 0$, $\lambda_{(\alpha,\beta)}[f]_h > 0$ and $\lambda_{(\varphi,\beta)}[g] > 0$ where $\exp(\varphi(\beta^{-1}(r))) < r$. Then*

$$\limsup_{r\to+\infty} \frac{\alpha(M_h^{-1}(M_{f(g)}(r)))}{\alpha(M_h^{-1}(M_f(r)))} = +\infty.$$

The Proof of Theorem 3.12 is omitted as it can be carried out in the line of Theorem 3.11 and in view of Theorem 3.10.

Two entire functions f and g are said to be asymptotically equivalent if there exists a, $0 < a < \infty$ such that $\frac{M_f(r)}{M_g(r)} \to a$ as $r \to +\infty$ and in this case we write $f \sim g$. If $f \sim g$, then clearly $g \sim f$. Our next theorem deals with the asymptotic behaviour of two entire functions.

Theorem 3.13. *Let f and g be any two entire functions such that $0 < \lambda_{(\alpha,\beta)}[f]_g \leq \rho_{(\alpha,\beta)}[f]_g < +\infty$. If $g \sim h$, then $\rho_{(\alpha,\beta)}[f]_h = \rho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[f]_h = \lambda_{(\alpha,\beta)}[f]_g$.*

Proof. Let $\varepsilon > 0$. Since $g \sim h$, then for any a $(0 < a < \infty)$ it follows for all sufficiently large positive numbers of r that

$$M_g(r) < (a+\varepsilon)M_h(r).$$

Now for $b > \max\{1, (a+\varepsilon)\}$, we get by Lemma 2.2 and the above that for all sufficiently large positive values of r

$$M_h^{-1}(r) < bM_g^{-1}(r).$$

Therefore we obtain that

$$\rho_{(\alpha,\beta)}[f]_h = \limsup_{r\to+\infty} \frac{\alpha(M_h^{-1}(M_f(r)))}{\beta(r)} \leq \limsup_{r\to+\infty} \frac{\alpha(bM_g^{-1}(M_f(r)))}{\beta(r)}.$$

Now we get from the above that $\rho_{(\alpha,\beta)}[f]_h \leq \rho_{(\alpha,\beta)}[f]_g$. The reverse inequality is clear because $h \sim g$ and so $\rho_{(\alpha,\beta)}[f]_g = \rho_{(\alpha,\beta)}[f]_h$.

In a similar manner, $\lambda_{(\alpha,\beta)}[f]_h = \lambda_{(\alpha,\beta)}[f]_g$.

This proves the theorem. □

Theorem 3.14. *Let f and g be any two entire functions such that $0 < \lambda_{(\alpha,\beta)}[f]_g \leq \rho_{(\alpha,\beta)}[f]_g < +\infty$. If $f \sim h$, then $\rho_{(\alpha,\beta)}[h]_g = \rho_{(\alpha,\beta)}[f]_g$ and $\lambda_{(\alpha,\beta)}[h]_g = \lambda_{(\alpha,\beta)}[f]_g$.*

Proof. Since $f \sim h$, then for any $\varepsilon > 0$ we obtain that

$$M_f(r) < (a+\varepsilon)M_h(r),$$

where $0 < a < \infty$.

Therefore for $b > \max\{1, (a+\varepsilon)\}$ and in view of Lemma 2.2, we get from above for all sufficiently large positive numbers of r that

$$M_f(r) < M_h(br).$$

Now we obtain from the above that

$$i.e.,\ \rho_{(\alpha,\beta)}[f]_g = \limsup_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\beta(r)} \leq \limsup_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_h(br)))}{\beta(r)}.$$

Now we get from the above that $\rho_{(\alpha,\beta)}[f]_g \leq \rho_{(\alpha,\beta)}[h]_g$. Further $f \sim h \Rightarrow h \sim f$, so we also obtain that $\rho_{(\alpha,\beta)}[h]_g \leq \rho_{(\alpha,\beta)}[f]_g$ and therefore $\rho_{(\alpha,\beta)}[h]_g = \rho_{(\alpha,\beta)}[f]_g$.

In a similar manner, $\lambda_{(\alpha,\beta)}[h]_g = \lambda_{(\alpha,\beta)}[f]_g$.

This proves the theorem. □

Theorem 3.15. *Let f and g be any two entire functions such that $0 < \lambda_{(\alpha,\beta)}[g]_f \leq \rho_{(\alpha,\beta)}[g]_f < +\infty$. Also let k and h be any two entire functions such that $g \sim h$ and $f \sim k$. Then $\rho_{(\alpha,\beta)}[g]_f = \rho_{(\alpha,\beta)}[h]_k = \rho_{(\alpha,\beta)}[h]_f = \rho_{(\alpha,\beta)}[g]_k$ and $\lambda_{(\alpha,\beta)}[g]_f = \lambda_{(\alpha,\beta)}[h]_k = \lambda_{(\alpha,\beta)}[h]_f = \lambda_{(\alpha,\beta)}[g]_k$.*

Theorem 3.15 follows from Theorem 3.13 and Theorem 3.14.

Now we state the following four theorems which can easily be carried out from the definitions of the generalized relative order (α,β) and the generalized relative lower order (α,β) of an entire function with respect to another entire function and with the help of Theorem 3.13, Theorem 3.14 and Theorem 3.15, and therefore their proofs are omitted.

Theorem 3.16. *Let f, g and h be any three entire functions such that $g \sim h$, $0 < \lambda_{(\alpha,\beta)}[f]_g \leq \rho_{(\alpha,\beta)}[f]_g < +\infty$ and $0 < \lambda_{(\alpha,\beta)}[f]_h \leq \rho_{(\alpha,\beta)}[f]_h < +\infty$. Then*

$$\liminf_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\alpha(M_h^{-1}(M_f(r)))} \leq 1 \leq \limsup_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\alpha(M_h^{-1}(M_f(r)))}.$$

Theorem 3.17. *Let f, g and h be any three entire functions such that $f \sim h$, $0 < \lambda_{(\alpha,\beta)}[f]_g \leq \rho_{(\alpha,\beta)}[f]_g < +\infty$ and $0 < \lambda_{(\alpha,\beta)}[h]_g \leq \rho_{(\alpha,\beta)}[h]_g < +\infty$. Then*

$$\liminf_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\alpha(M_g^{-1}(M_h(r)))} \leq 1 \leq \limsup_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\alpha(M_g^{-1}(M_h(r)))}.$$

Theorem 3.18. *Let f, g, h and k be any four entire functions such that $f \sim h$ and $g \sim k$. Also let $0 < \lambda_{(\alpha,\beta)}[f]_g \leq \rho_{(\alpha,\beta)}[f]_g < +\infty$ and $0 < \lambda_{(\alpha,\beta)}[h]_k \leq \rho_{(\alpha,\beta)}[h]_k < +\infty$. Then*

$$\liminf_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\alpha(M_k^{-1}(M_h(r)))} \leq 1 \leq \limsup_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_f(r)))}{\alpha(M_k^{-1}(M_h(r)))}.$$

Theorem 3.19. *Let f, g, h and k be any four entire functions such that $f \sim h$ and $g \sim k$. Also let $0 < \lambda_{(\alpha,\beta)}[h]_g \leq \rho_{(\alpha,\beta)}[h]_g < +\infty$ and $0 < \lambda_{(\alpha,\beta)}[f]_k \leq \rho_{(\alpha,\beta)}[f]_k < +\infty$. Then*

$$\liminf_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_h(r)))}{\alpha(M_k^{-1}(M_f(r)))} \leq 1 \leq \limsup_{r\to+\infty} \frac{\alpha(M_g^{-1}(M_h(r)))}{\alpha(M_k^{-1}(M_f(r)))}.$$

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