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ON THE VALUE SHARING OF q-c-SHIFT AND q-SHIFT MONOMIALS OF MEROMORPHIC FUNCTIONS AND THEIR DERIVATIVES

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ABSTRACT. In this paper, we employ the notion of weighted sharing to study the uniqueness problems of q-c-shift and q-shift monomials of transcendental meromorphic functions of zero order sharing 1-points. The results in this paper extend some previous results.

1. INTRODUCTION AND RESULTS

We use the standard notation and fundamental results of Nevanlinna theory (see [7, 13, 18]) and by meromorphic functions we will always mean meromorphic functions in the complex plane. For a non-constant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic function of f. We define $\alpha(z) \not\equiv 0,\infty$ as a small function with respect to $f(z)$, if $T(r,\alpha) = S(r,f)$, where $S(r,f)$ denotes any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. The order of f is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and a be any complex constant. We say that $f(z)$ and $g(z)$ share the value a CM (counting multiplicities) if $f(z)-a$ and $g(z)-a$ have the same zeros with the same multiplicities and $f(z)$, $g(z)$ share a IM (ignoring multiplicities) if only the locations of zeros are considered.

Around 2001, I. Lahiri introduced the concept of weighted sharing in the literature [11, 12]. It indicates the gradual change of shared values from CM to IM. We recall the definition below.

Definition 1.1. *[12] Let $k \in \mathbb{N}\cup\{0\}\cup\{\infty\}$ and $a \in \mathbb{C}\cup\{\infty\}$. We denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \le k$ and k+1 times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.*

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Clearly if f, g share (a,k) then f, g share (a,p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

In 2006, Halburd-Korhonen [6] obtained the difference analogue of the logarithmic derivative lemma for a finite order meromorphic function. In the next year, the same type of result corresponding to $f(qz)$ for zero-order meromorphic function was discovered in [4]. These two results induced great interest among the researchers to investigate the uniqueness problem of entire or meromorphic functions and their shift or difference operator.

For $q \in \mathbb{C}\backslash\{0,1\}$, shift, q-c-shift and q-shift operators of a non-constant meromorphic function are defined by $f(z+c)$, $f(qz+c)$ and $f(qz)$ respectively.

In this paper, by $P(z)$ we mean the polynomial: $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where $a_0, a_1, \ldots, a_n \neq 0$ are complex constants and $n(\geq 1)$ is an integer. For a meromorphic function h and a finite complex constant c, we define $P(h)(z)h(qz+c)$ and $P(h)(z)h(qz)$ as q-c-shift and q-shift monomials respectively. For the sake of convenience, let $\Gamma_0 = m_1 + m_2$ and $\Gamma_1 = m_1 + 2m_2$, where m_1, m_2 respectively is the number of simple and multiple zeros of $P(z)$.

In 2013, the first theorem on the q-c-shift operator was presented by Lui-Cao-Qi-Yi [15] as follows:

Theorem A. *[15] Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of zero order. Suppose that q and c are two non-zero complex constants and $n \in \mathbb{N}$ is such that $f^n(z)f(qz+c)$ and $g^n(z)g(qz+c)$ share $(1,l)$.*

(i) *If $l = \infty$ and $n \geq 14$ or*
(ii) *if $l = 0$ and $n \geq 26$,*
then $f(z) \equiv tg(z)$ or $f(z)g(z) \equiv t$ for some constants t that satisfy $t^{n+1} = 1$.

In the same year, Huang [10] studied the analogous result considering q-shift operator for CM sharing while Qi-Yang [16] supplemented the same for IM sharing.

Theorem B. *[10, 16] Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions of zero order. Suppose that q is a non-zero complex constant and $n \in \mathbb{N}$ is such that $f^n(z)f(qz)$ and $g^n(z)g(qz)$ share $(1,l)$.*

(i) *If $l = \infty$ and $n \geq 14$ or*
(ii) *if $l = 0$ and $n \geq 26$,*
then $f(z) \equiv tg(z)$ or $f(z)g(z) \equiv t$ for some constants t that satisfy $t^{n+1} = 1$.

In addition, Qi-Yang [16] studied a different form of q shift monomial as follows:

Theorem C. *[16] Let $f(z)$, $g(z)$ be two transcendental meromorphic functions of zero order. Suppose that q is a non-zero complex constant such that $|q| \neq 1$ and r is a positive integer satisfying $r \geq 30$ such that $f(z)^r(f(z)-1)f(qz)$ and $g(z)^r(g(z)-1)g(qz)$ share $(1,0)$, $f(z)$ and $g(z)$ share $(\infty,0)$, then $f(z)^r(f(z)-1)f(qz) = g(z)^r (g(z)-1)g(qz)$.*

In 2015, Zhao-Zhang [21] considered the derivative counterpart of *Theorem A* in the following manner.

Theorem D. *[21] Let $f(z)$ and $g(z)$ be two transcendental entire functions of zero order. Suppose that q, c are two non-zero complex constants and $n\in\mathbb{N}$ is such that $(f^n(z)f(qz+c))^{(k)}$ and $(g^n(z)g(qz+c))^{(k)}$ share $(1,l)$.*

(i) *If $l=\infty$ and $n>2k+5$ or*

(ii) *if $l=0$ and $n>5k+11$,*

then $f(z)\equiv tg(z)$ or $f(z)g(z)\equiv t$ for some constants t that satisfy $t^{n+1}=1$.

As far as our knowledge is concerned, no such attempt has yet been made to employ the notion of weighted sharing in the field of q-c-shift and q-shift operators. Since the lower bound of n or r in the above theorems can not further be reduced in case of CM sharing, the only possibility for improvement is to relax the sharing constrains. So, we have manipulated the notion of weight sharing to relax the CM sharing results keeping the lower bound of n or r the same. Actually the purpose of the present paper is to improve all the *Theorems A-D* in terms of the most generalized form of the monomials. The following theorem is an extension of *Theorems A* and *B*.

Theorem 1.1. *Let $f(z)$, $g(z)$ be two transcendental meromorphic functions of zero order, $c\in\mathbb{C}$. Suppose that $F=P(f)(z)f(qz+c)$ and $G=P(g)(z)g(qz+c)$ share $(1,l)$. Now*

(i) *if $l\geq 2$ and $n>2\Gamma_1+9$ or*

(ii) *if $l=1$ and $n>2\Gamma_1+\frac{1}{2}\Gamma_0+\frac{21}{2}$ or*

(iii) *if $l=0$ and $n>2\Gamma_1+3\Gamma_0+18$,*

then either

$$P(f)(z)f(qz+c).P(g)(z)g(qz+c)\equiv 1$$

or

$$P(f)(z)f(qz+c)\equiv P(g)(z)g(qz+c).$$

In particular, for any integer $n\geq 1$, we consider $P(f)=f^n$ and

(i) *if $l\geq 2$ and $n\geq 14$ or*

(ii) *if $l=1$ and $n\geq 16$ or*

(iii) *if $l=0$ and $n\geq 26$,*

then either $f\equiv tg$ or $fg\equiv t$, for some constant t such that $t^{n+1}=1$.

Remark 1.1. Conclusions (i) and (iii) under $P(f)=f^n$ in *Theorem 1.1*, yield *Theorems A* and *B* for the case $c\neq 0$ and $c=0$, respectively. Therefore, *Theorem 1.1* is a huge extension of *Theorem A* and *B*, in the direction of the general polynomial $P(f)$ as well as the relaxation of sharings.

In the next theorems we shall show that when $c=0$, the conclusion of *Theorem 1.1*, becomes more precise. However, the same is possible for a particular form of $P(f)$ namely $P(f)=f^r(z)(f^m(z)-1)^p$, where r, m, p be any positive integers.

At first we deal the case when $p(\geq 2)$ is any positive integer and the following theorem is an improvement of *Theorem C*.

Theorem 1.2. *Let $f(z)$, $g(z)$ be two transcendental meromorphic functions of zero order and q be a non-zero complex constant such that $|q| \neq 1$. Suppose r is an integer such that $f^r(z)(f^m(z)-1)^p f(qz)$ and $g^r(z)(g^m(z)-1)^p g(qz)$ share $(1,l)$, $f(z)$ and $g(z)$ share $(\infty,0)$. Now*

(i) *if $l \geq 2$ and $r > 4m - mp + 13$ or*
(ii) *if $l = 1$ and $r > \frac{9}{2}m - mp + 15$ or*
(iii) *if $l = 0$ and $r > 7m - mp + 25$,*
then $f^r(z)(f^m(z)-1)^p f(qz) = g^r(z)(g^m(z)-1)^p g(qz)$.

In *Theorem 1.2*, putting $m = 1$, we can easily derive the following corollary.

Corollary 1.1. *Let $f(z)$, $g(z)$ be two transcendental meromorphic functions of zero order and q be a non-zero complex constant such that $|q| \neq 1$. Suppose r is an integer such that $f^r(z)(f(z)-1)^p f(qz)$ and $g^r(z)(g(z)-1)^p g(qz)$ share $(1,l)$, $f(z)$ and $g(z)$ share $(\infty,0)$. Now*

(i) *if $l \geq 2$ and $r > 17 - p$ or*
(ii) *if $l = 1$ and $r > \frac{39}{2} - p$ or*
(iii) *if $l = 0$ and $r > 32 - p$,*
then $f^r(z)(f(z)-1)^p f(qz) = g^r(z)(g(z)-1)^p g(qz)$.

The next example shows that one cannot get $f(z) \equiv g(z)$ from $f^r(z)(f(z)-1)^p f(qz) \equiv g^r(z)(g(z)-1)^p g(qz)$ for $p \geq 1$ if $f(z)$ and $g(z)$ are non-constant meromorphic functions, even if $f(z)$ and $g(z)$ share (∞,∞).

Example 1.1. *Let q be a constant $(|q| \neq 0,1)$, n be a positive integer. Suppose that*

$$f(z) = \frac{H^{r+p}(z)H(qz) - H^p(z)}{H^{r+p}(z)H(qz) - 1},$$
$$g(z) = \frac{H^r(z)H(qz) - 1}{H^{r+p}(z)H(qz) - 1},$$

where $H(z)$ is a non-constant entire function (can be a non-constant polynomial) of zero-order. Clearly, $f^r(z)(f(z)-1)^p f(qz)$ and $g^r(z)(g(z)-1)^p g(qz)$ share $(1,\infty)$ and $f(z)$ and $g(z)$ share (∞,∞). Moreover, $f^r(z)(f(z)-1)^p f(qz) \equiv g^r(z)(g(z)-1)^p g(qz)$, but $f(z) \not\equiv g(z)$.

Next we turn our attention to the case $p = 1$. Thus we get a counterpart of *Theorem 1.2*, which improves *Theorem C*.

Theorem 1.3. *Let $f(z)$, $g(z)$ be two transcendental meromorphic functions of zero order and q be a non-zero complex constant such that $|q| \neq 1$. Suppose r is an integer such that $f^r(z)(f^m(z)-1)f(qz)$ and $g^r(z)(g^m(z)-1)g(qz)$ share $(1,l)$, $f(z)$ and $g(z)$ share $(\infty,0)$. Now*

(i) *if* $l \geq 2$ *and* $r > m + 13$ *or*
(ii) *if* $l = 1$ *and* $r > \frac{3}{2}m + 15$ *or*
(iii) *if* $l = 0$ *and* $r > 4m + 25$,
then $f^r(z)(f^m(z) - 1)f(qz) = g^r(z)(g^m(z) - 1)g(qz)$.

From *Theorem 1.3*, taking $m = 1$, we can easily deduce the following corollary.

Corollary 1.2. *Let* $f(z)$, $g(z)$ *be two transcendental meromorphic functions of zero order and* q *be a non-zero complex constant such that* $|q| \neq 1$. *Suppose* r *is an integer such that* $f^r(z)(f(z) - 1)f(qz)$ *and* $g^r(z)(g(z) - 1)g(qz)$ *share* $(1, l)$, $f(z)$ *and* $g(z)$ *share* $(\infty, 0)$. *Now*
(i) *if* $l \geq 2$ *and* $r \geq 15$ *or*
(ii) *if* $l = 1$ *and* $r \geq 18$ *or*
(iii) *if* $l = 0$ *and* $r \geq 30$,
then $f^r(z)(f(z) - 1)f(qz) = g^r(z)(g(z) - 1)g(qz)$.

Remark 1.2. Note that from *(iii)* of *Corollary 1.2*, we get *Theorem C* again. Hence, in view of the generalized polynomial $P(f)$, and relaxation of sharings, *Theorems 1.2, 1.3* and *Corollaries 1.1, 1.2* are great extensions of *Theorem C*.

The next theorem is an extension of *Theorem D* for meromorphic functions.

Theorem 1.4. *Let* $f(z)$ *and* $g(z)$ *be transcendental meromorphic functions of zero order. Suppose* q *is a non-zero complex constant and* $c \in \mathbb{C}$ *and* n *is an integer such that* $(P(f)f(qz+c))^{(k)}$ *and* $(P(g)g(qz+c))^{(k)}$ *share* $(1, l)$; $l = 0, 1, 2$. *Now*
(i) *if* $l \geq 2$ *and* $n > 2(m_2 + 1)k + 2\Gamma_1 + 9$ *or*
(ii) *if* $l = 1$ *and* $n > \left(\frac{5}{2}m_2 + 3\right)k + 2\Gamma_1 + \frac{1}{2}\Gamma_0 + \frac{21}{2}$ *or*
(iii) *if* $l = 0$ *and* $n > (5m_2 + 8)k + 2\Gamma_1 + 3\Gamma_0 + 18$,
then one of the following results hold:
(1) $(P(f)f(qz+c))^{(k)} . (P(g)g(qz+c))^{(k)} = 1$ *or*
(2) $f(z) \equiv tg(z)$ *for a constant* t *such that* $t^\lambda = 1$, *where* λ *is the GCD of the elements of* J, $J = \{k+1 \in I : a_k \neq 0\}$ *and* $I = \{1, 2, \ldots, n+1\}$. *In particular* $P(z) = a_n z^n$, $f \equiv tg$ *for a constant* t *such that* $t^{n+1} = 1$ *or*
(3) f *and* g *satisfy algebraic equation* $R(f(z), g(z)) = 0$, *where* $R(w_1, w_2) = P(w_1)w_1(qz+c) - P(w_2)w_2(qz+c)$.

Corollary 1.3. *Under the same assumptions as in Theorem 1.4, if* $f(z)$ *and* $g(z)$ *are transcendental entire functions of zero order and*
(i) *if* $l \geq 2$ *and* $n > 2\Gamma_1 + 2km_2 + 1$ *or*
(ii) *if* $l = 1$ *and* $n > 2\Gamma_1 + \frac{1}{2}\Gamma_0 + \frac{5}{2}km_2 + \frac{3}{2}$ *or*
(iii) *if* $l = 0$ *and* $n > 2\Gamma_1 + 3\Gamma_0 + 5km_2 + 4$,
then one of the following results holds:

(1) *$f(z) \equiv tg(z)$ for a constant t such that $t^{\lambda} = 1$, where λ is the GCD of the elements of J, $J = \{k+1 \in I : a_k \neq 0\}$ and $I = \{1, 2, \ldots, n+1\}$. In particular $P(z) = a_n z^n$, $f \equiv tg$ for a constant t such that $t^{n+1} = 1$ or*

(2) *f and g satisfy algebraic equation $R(f(z), g(z)) = 0$, where $R(w_1, w_2) = P(w_1)w_1(qz+c) - P(w_2)w_2(qz+c)$.*

Remark 1.3. If we put, $P(f) = f^n$, $\Gamma_1 = 2$, $\Gamma_0 = 1$ and $m_2 = 1$ in *Corollary 1.3*, then from *(i)* and *(iii)*, we have *Theorem D*. So, *Corollary 1.3* is an improvement as well as an extension of *Theorem D*.

2. Auxiliary Definitions

Throughout the paper we have used the following definitions and notations.

Definition 2.1. *[8] Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid = 1)$ the counting function of simple a points of f. For $p \in \mathbb{N}$ we denote by $N(r, a; f \mid \leq p)$ the counting function of those a-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, a; f \mid \leq p)$ we denote the corresponding reduced counting function.*

In an analogous manner we can define $N(r, a; f \mid \geq p)$ and $\overline{N}(r, a; f \mid \geq p)$.

Definition 2.2. *[12] Let $p \in \mathbb{N} \cup \{\infty\}$. We denote by $N_p(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq p$ and p times if $m > p$. Then $N_p(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid \geq 2) + \ldots + \overline{N}(r, a; f \mid \geq p)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.*

Definition 2.3. *[19] Let f and g be two non-constant meromorphic functions such that f and g share $(a, 0)$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ the reduced counting function of those a-points of f and g where $p > q$, by $N_E^{1)}(r, a; f)$ the counting function of those a-points of f and g where $p = q = 1$, by $\overline{N}_E^{(2}(r, a; f)$ the reduced counting function of those a-points of f and g where $p = q \geq 2$. In the same way we can define $\overline{N}_L(r, a; g)$, $N_E^{1)}(r, a; g)$, $\overline{N}_E^{(2}(r, a; g)$. In a similar manner we can define $\overline{N}_L(r, a; f)$ and $\overline{N}_L(r, a; g)$ for $a \in \mathbb{C} \cup \{\infty\}$.*

When f and g share (a, m), $m \geq 1$, then $N_E^{1)}(r, a; f) = N(r, a; f \mid = 1)$.

Definition 2.4. *[11, 12] Let f, g share a value $(a, 0)$. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.*

Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

3. Lemmas

For two non-constant meromorphic functions F and G, in what follows H represents the following function.

$$H=\left(\frac{F^{''}}{F^{'}}-\frac{2F^{'}}{F-1}\right)-\left(\frac{G^{''}}{G^{'}}-\frac{2G^{'}}{G-1}\right). \tag{3.1}$$

Lemma 3.1. *[17] Let f be a zero order meromorphic function and $q\in\mathbb{C}\backslash\{0\}$, $c\in\mathbb{C}$. Then*

$$m\left(r,\frac{f(z)}{f(qz+c)}\right)=S(r,f)$$

and

$$T(r,f(qz+c))=T(r,f)+S(r,f).$$

Lemma 3.2. *[5] If $T:\mathbb{R}^+\to\mathbb{R}^+$ is an increasing function such that*

$$\limsup_{r\to\infty}\frac{\log T(r)}{\log r}=0,$$

then the set

$$E=\{r\mid T(C_1r)\geq C_2T(r)\}$$

has logarithmic density 0 for all $C_1>1$ and $C_2>1$.

Lemma 3.3. *[9, Theorems 6 and 7] Let $f(z)$ be a meromorphic function of finite order and let $c\neq 0$ be fixed. Then*

$$\begin{aligned}N(r,0;f(z+c))&\leq N(r,0;f(z))+S(r,f),\\ N(r,\infty;f(z+c))&\leq N(r,\infty;f)+S(r,f),\\ \overline{N}(r,0;f(z+c))&\leq \overline{N}(r,0;f(z))+S(r,f),\\ \overline{N}(r,\infty;f(z+c))&\leq \overline{N}(r,\infty;f)+S(r,f).\end{aligned}$$

Lemma 3.4. *Let f be a non-constant meromorphic function of finite order and $q\in\mathbb{C}\backslash\{0\}$, $c\in\mathbb{C}$. Then*

$$\begin{aligned}N(r,0;f(qz+c))&\leq N(r,0;f(z))+S(r,f),\\ N(r,\infty;f(qz+c))&\leq N(r,\infty;f)+S(r,f),\\ \overline{N}(r,0;f(qz+c))&\leq \overline{N}(r,0;f(z))+S(r,f),\\ \overline{N}(r,\infty;f(qz+c))&\leq \overline{N}(r,\infty;f)+S(r,f).\end{aligned}$$

Proof. First we consider the case $|q|\geq 1$. By a simple geometric observation, we obtain

$$N(r,0;f(qz+c))\leq N(|q|r,0;f(z+(c/q))). \tag{3.2}$$

Since f is of order 0, then from *Lemma 3.2*, we have

$$\begin{aligned}N(|q|r,0;f(z+(c/q)))&\leq N(r,0;f(z+(c/q)))+S(r,f)\\ &\leq N(|q|r,0;f(z+(c/q)))+S(r,f)\\ \Longrightarrow N(|q|r,0;f(z+(c/q)))&=N(r,0;f(z+(c/q)))+S(r,f)\end{aligned} \tag{3.3}$$

on a set of logarithmic density 1.

From (3.2) and (3.3), we have

$$N(r,0;f(qz+c)) \leq N(r,0;f(z+(c/q)))+S(r,f).$$

For $c=0$, the first inequality is obvious. Next for $c\neq 0$, using the first inequality of *Lemma 3.3*, we have

$$\begin{aligned} N(r,0;f(qz+c)) &\leq N(r,0;f(z+(c/q)))+S(r,f)\\ &\leq N(r,0;f(z))+S(r,f). \end{aligned}$$

Next for the case $|q|\leq 1$, in a similar way, we can prove this.
Similarly, adopting the same method we can prove the other three inequalities. □

Lemma 3.5. *[20] Let F, G be two non-constant meromorphic functions sharing $(1,0)$ and $H\not\equiv 0$. Then*

$$N_E^{1)}(r,1;F)=N_E^{1)}(r,1;G)\leq N(r,H)+S(r,F)+S(r,G).$$

Lemma 3.6. *If two non-constant meromorphic functions F and G share $(1,0)$ and $H\not\equiv 0$, then*

$$\begin{aligned} \overline{N}(r,\infty;H) &\leq \overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}(r,\infty;F\mid\geq 2)\\ &\quad+\overline{N}(r,\infty;G\mid\geq 2)+\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G'), \end{aligned}$$

where by $\overline{N}_0(r,0;F')$ we mean the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Proof. The proof can be carried out in the line of the proof of [12, Lemma 2]. One can easily verify that possible poles of H occur at (i) multiple zeros of f and g, (ii) multiple poles of f and g, (iii) those 1-points of f and g whose multiplicities are distinct from the multiplicities of the corresponding 1-points of g and f, respectively, (iv) zeros of f' which are not the zeros of $f(f-1)$ and (v) zeros of g' which are not zeros of $g(g-1)$. Since H has only simple poles, the lemma follows from the above. This proves the lemma. □

Lemma 3.7. *[3] Let f, g be two non-constant meromorphic functions sharing $(1,l)$, where $0\leq l<\infty$. Then*

$$\begin{aligned} &\overline{N}(r,1;f)+\overline{N}(r,1;g)-N_E^{1)}(r,1;f)+\left(l-\frac{1}{2}\right)\overline{N}_*(r,1;f,g)\\ &\qquad\leq \frac{1}{2}[N(r,1;f)+N(r,1;g)]. \end{aligned}$$

Lemma 3.8. *Let f and g be any two meromorphic function and suppose they share $(1,l)$. Then*

$$\overline{N}_*(r,1;f,g)\leq\frac{1}{l+1}\left[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\right]+S(r,f)+S(r,g).$$

Proof. In view of *Definition 2.4*, using Lemma 2.14 [1], we proceed as follows:

$$
\begin{aligned}
\overline{N}_*(r,a;f,g) &= \overline{N}_L(r,a;f)+\overline{N}_L(r,a;g)\\
&\leq \frac{1}{l+1}\left[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\right.\\
&\quad \left. - N_{\otimes}(r,0;f')-N_{\otimes}(r,0;g')\right]+S(r,f)+S(r,g)\\
&\leq \frac{1}{l+1}\left[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;g)\right]+S(r,f)+S(r,g),
\end{aligned}
$$

where $N_{\otimes}(r,0;f') = N(r,0;f'|f \neq 0,\omega_1,\ldots,\omega_n)$ such that $\omega_1,\ldots,\omega_n$ are the distinct roots of the equation $z^n + az^{n-1} + b = 0$. □

Lemma 3.9. *[12] Let f, g be two non-constant meromorphic functions sharing $(1,2)$. Then one of the following cases holds:*

(i) $T(r,f) \leq N_2(r,0;f)+N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)+S(r,f)+S(r,g)$, *the same inequality holds for T(r, g);*

(ii) $f = g$;

(iii) $f \cdot g = 1$.

Lemma 3.10. *[2] Let f, g be two transcendental meromorphic functions sharing $(1,1)$ and $H \not\equiv 0$, then*

$$
\begin{aligned}
T(r,f) \leq\ & N_2(r,0;f)+N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)\\
&+\frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)+S(r,f)+S(r,g).
\end{aligned}
$$

Lemma 3.11. *[2] Let f, g be two transcendental meromorphic functions sharing $(1,1)$ and $H \not\equiv 0$, then*

$$
\begin{aligned}
T(r,f) \leq\ & N_2(r,0;f)+N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)+2\overline{N}(r,0;f)\\
&+2\overline{N}(r,\infty;f)+2\overline{N}(r,0;g)+2\overline{N}(r,\infty;g)+S(r,f)+S(r,g).
\end{aligned}
$$

Lemma 3.12. *[14] Let f be a non-constant meromorphic function and let p and k be two positive integers. Then*

$$
\begin{aligned}
N_p\left(r,\frac{1}{f^{(k)}}\right) &\leq T(r,f^{(k)})-T(r,f)+N_{p+k}\left(r,\frac{1}{f}\right)+S(r,f);\\
N_p\left(r,\frac{1}{f^{(k)}}\right) &\leq k\overline{N}(r,f)+N_{p+k}\left(r,\frac{1}{f}\right)+S(r,f).
\end{aligned}
$$

Lemma 3.13. *Let $f(z)$ be a transcendental meromorphic function of finite order. Then for $n > 1$ we have*

$$(n-1)\,T(r,f) \leq T(r,P(f)(z)f(qz+c))+S(r,f).$$

Proof. By *Lemma 3.1*, we get

$$
\begin{aligned}
nT(r,f) &= T(r,P(f)(z))+O(1)\\
&= T\left(r,P(f)(z)f(qz+c)\frac{1}{f(qz+c)}\right)+O(1)
\end{aligned}
$$

$$\leq T(r,P(f)(z)f(qz+c))+T\left(r,\frac{1}{f(qz+c)}\right)+O(1)$$
$$=T(r,P(f)(z)f(qz+c))+T(r,f)+S(r,f).$$

So,

$$(n-1)T(r,f)\leq T(r,P(f)(z)f(qz+c))+S(r,f).$$

This completes the proof of the lemma. □

Lemma 3.14. *Let $f(z)$ be a transcendental entire function of finite order. Then for $n>1$ we have*

$$T(r,P(f)(z)f(qz+c))=(n+1)T(r,f)+S(r,f).$$

Proof. By *Lemma 3.1*, we get

$$\begin{aligned}T(r,P(f)(z)f(qz+c))&\leq T(r,P(f))+T(r,f(qz+c))+O(1)\\&\leq(n+1)T(r,f)+S(r,f)\end{aligned}$$

and

$$\begin{aligned}(n+1)T(r,f)&\leq T(r,P(f)(z)f(z))+O(1)\\&=m(r,P(f)(z)f(z))+O(1)\\&\leq m(r,P(f)(z)f(qz+c))+m\left(r,\frac{f(z)}{f(qz+c)}\right)+O(1)\\&\leq m(r,P(f)(z)f(qz+c))+S(r,f)\\&=T(r,P(f)(z)f(qz+c))+S(r,f).\end{aligned}$$

This completes the proof of the lemma. □

4. Proofs of the Theorems

Proof of Theorem 1.1. The proof of the theorem is based on the ideas in [Theorems 1, 2; [15]]. Here we consider $F(z)=P(f)(z)f(qz+c)$ and $G(z)=P(g)(z)g(qz+c)$. Then F and G share $(1,l)$.

Case-1 Let $H\not\equiv 0$. Using *Lemmas 3.5* and *3.6*, we have

$$\begin{aligned}N_E^{1)}(r,1;F)&\leq N(r,H)+S(r,F)+S(r,G)\\&\leq\overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}(r,\infty;F\mid\geq 2)+\overline{N}(r,\infty;G\mid\geq 2)\\&\quad+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+\overline{N}_*(r,1;F,G).\end{aligned}\tag{4.1}$$

By the second fundamental theorem, we get

$$T(r,F)\leq\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)-\overline{N}_0(r,0;F^{'})+S(r,f)\tag{4.2}$$

and

$$T(r,G)\leq\overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}(r,1;G)-\overline{N}_0(r,0;G^{'})+S(r,g).\tag{4.3}$$

Combining (4.1), (4.2) and (4.3) with the help of *Lemmas 3.7* and *3.8*, we have

$$
\begin{aligned}
&[T(r,F)+T(r,G)]\\
&\le [\overline{N}(r,0;F)+\overline{N}(r,0;G)]+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)]\\
&\quad +[\overline{N}(r,1;F)+\overline{N}(r,1;G)]-[\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})]+S(r,f)+S(r,g)\\
&\le N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\\
&\quad +[\overline{N}(r,1;F)+\overline{N}(r,1;G)-N_E^{1)}(r,1;F)]+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\le N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\\
&\quad +\frac{1}{2}[T(r,F)+T(r,G)]-\left(l-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\le \frac{1}{2}[T(r,F)+T(r,G)]+N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)\\
&\quad +N_2(r,\infty;G)+\frac{(3-2l)}{2(l+1)}[\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)\\
&\quad +\overline{N}(r,\infty;G)]+S(r,f)+S(r,g) \qquad (4.4)
\end{aligned}
$$

$$
\begin{aligned}
&\Longrightarrow [T(r,F)+T(r,G)]\\
&\le 2\left[N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\right]+\frac{(3-2l)}{(l+1)}\left[\overline{N}(r,0;F)\right.\\
&\quad \left.+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)\right]+S(r,f)+S(r,g). \qquad (4.5)
\end{aligned}
$$

Subcase 1.1. While $l \ge 2$, in view of *Lemmas 3.4* and *3.13*, from (4.5) we get

$$
\begin{aligned}
&(n-1)[T(r,f)+T(r,g)]\\
&\le 2[(m_1+2m_2)T(r,f)+N(r,0;f(qz+c))+(m_1+2m_2)T(r,g)\\
&\quad +N(r,0;g(qz+c))+2\overline{N}(r,\infty;f)+N(r,\infty;f(qz+c))+2\overline{N}(r,\infty;g)\\
&\quad +N(r,\infty;g(qz+c))]+S(r,f)+S(r,g)\\
&\le 2\,[m_1+2m_2+4]\,\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \qquad (4.6)
\end{aligned}
$$

From (4.6) it follows that

$$
\begin{aligned}
&(n-1)[T(r,f)+T(r,g)]\\
&\le [2\Gamma_1+8]\,\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),
\end{aligned}
$$

which is a contradiction for $n > 2\Gamma_1+9$.

Subcase 1.2. While $l = 1$, using *Lemmas 3.4* and *3.13*, from (4.5) we get

$$
\begin{aligned}
&(n-1)[T(r,f)+T(r,g)]\\
&\le 2[(m_1+2m_2)T(r,f)+N(r,0;f(qz+c))+(m_1+2m_2)T(r,g)\\
&\quad +N(r,0;g(qz+c))+2\overline{N}(r,\infty;f)+N(r,\infty;f(qz+c))+2\overline{N}(r,\infty;g)
\end{aligned}
$$

$$+N(r,\infty;g(qz+c))]+\left(\frac{1}{2}\right)[(m_1+m_2)T(r,f)+N(r,0;f(z+c_j))$$
$$+2\overline{N}(r,\infty;f)+(m_1+m_2)T(r,g)+N(r,0;g(z+c_j))+2\overline{N}(r,\infty;g)]+S(r,f)+S(r,g)$$
$$\leq\left[2(m_1+2m_2+4)+\frac{1}{2}(m_1+m_2+3)\right]\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \quad (4.7)$$

From (4.7), it follows that

$$(n-1)[T(r,f)+T(r,g)]$$
$$\leq\left[2\Gamma_1+8+\frac{1}{2}(\Gamma_0+3)\right]\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$

which is a contradiction for $n>2\Gamma_1+\frac{1}{2}\Gamma_0+\frac{21}{2}$.

Subcase 1.3. Next let $l=0$. Again using *Lemmas 3.4* and *3.13*, from (4.5) we get

$$(n-1)[T(r,f)+T(r,g)]$$
$$\leq [2(m_1+2m_2+4)+3(m_1+m_2+3)]\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \quad (4.8)$$

From (4.8), we get

$$(n-1)[T(r,f)+T(r,g)]$$
$$\leq [2\Gamma_1+8+3(\Gamma_0+3)]\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$

which is a contradiction for $n>2\Gamma_1+3\Gamma_0+18$.

Case-2 Let $H\equiv 0$, integrating (3.1) we get

$$\frac{1}{F-1}\equiv\frac{bG+a-b}{G-1}, \quad (4.9)$$

where $a(\neq 0)$, b are constants. From (4.9) it is clear that F and G share $(1,\infty)$. Now we consider the following cases:

Case 1. Let $b\neq 0$ and $a\neq b$. If $b=-1$, from (4.9) we have

$$F\equiv\frac{-a}{G-a-1}.$$

From *Lemma 3.4*, we see that

$$\overline{N}(r,a+1;G)=\overline{N}(r,\infty;F)\leq 2\overline{N}(r,\infty;f).$$

So, in view of *Lemmas3.4* and *3.13*, using the second fundamental theorem, we get

$$\begin{aligned}(n-1)\,T(r,g) &\leq \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}(r,a+1;G)+S(r,g)\\ &\leq (m_1+m_2)T(r,g)+\overline{N}(r,0;g(z+c_j))+2(\overline{N}(r,\infty;g)\\ &\quad +\overline{N}(r,\infty;f))+S(r,g)\\ &\leq (m_1+m_2+3)T(r,g)+2T(r,f)+S(r,g).\end{aligned}$$

In a similar manner, we can get

$$(n-1)\,T(r,f)\leq(m_1+m_2+3)T(r,f)+2T(r,g)+S(r,f).$$

Combining the above two equations, we can get

$$(n-1)\{T(r,f)+T(r,g)\} \leq (\Gamma_0+5)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$

a contradiction for $n > 2\Gamma_1 + 9$.
If $b \neq -1$, from (4.9) we get

$$F-\left(1+\frac{1}{b}\right) \equiv \frac{-a}{b^2[G+\frac{a-b}{b}]}.$$

So,

$$\overline{N}\left(r, \frac{(b-a)}{b}; G\right) = \overline{N}(r,\infty;F).$$

Using *Lemmas 3.4*, *3.13* and with the same argument as used in the case for $b=-1$, we can get a contradiction.

Case 2. Let $b \neq 0$ and $a = b$. If $b = -1$, then from (4.9) we have

$$FG \equiv 1,$$

i.e.,

$$P(f)(z)f(qz+c)P(g)(z)g(qz+c) \equiv 1.$$

In particular, when $P(f) = f^n$, take $M(z) = f(z)g(z)$. When $M(z)$ is non-constant, we have from above

$$M^n(z) \equiv \frac{1}{M(qz+c)}.$$

So, using the first fundamental theorem and *Lemma 3.1*, we have

$$nT(r,M) = T(r,M(qz+c)) + O(1) = T(r,M) + S(r,M),$$

a contradiction. So, $M(z)$ must be a constant and $M(z)^{n+1} \equiv 1$, which implies $fg \equiv t$, where $t^{n+1} = 1$.
If $b \neq -1$, from (4.9) we have

$$\frac{1}{F} \equiv \frac{bG}{(1+b)G-1}.$$

Therefore,

$$\overline{N}\left(r, \frac{1}{1+b}; G\right) = \overline{N}(r,0;F).$$

So, in view of *Lemmas 3.4* and *3.13*, using the second fundamental theorem, we have

$$\begin{aligned}(n-1)\ T(r,g) &\leq \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}\left(r,\frac{1}{1+b};G\right)+S(r,g)\\ &\leq (m_1+m_2+3)T(r,g)+(m_1+m_2+1)T(r,f)+S(r,g).\end{aligned}$$

In a similar manner, we can get

$$(n-1)\ T(r,f) \leq (m_1+m_2+3)T(r,f)+(m_1+m_2+1)T(r,g)+S(r,f).$$

Combining the above two equations, we can get

$$(n-1)\{T(r,f)+T(r,g)\} \leq (2\Gamma_0+4)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$

a contradiction for $n > 2\Gamma_1 + 9$.

Case 3. Let $b = 0$. From (4.9), we obtain

$$F \equiv \frac{G+a-1}{a}. \tag{4.10}$$

If $a \neq 1$ then from (4.10) we obtain

$$\overline{N}(r,1-a;G) = \overline{N}(r,0;F).$$

Now using a similar process as done in Case 2, for $b \neq -1$, we can deduce a contradiction. Therefore $a = 1$ and from (4.10) we obtain $F \equiv G$, i.e.,

$$P(f)(z)f(qz+c) \equiv P(g)(z)g(qz+c).$$

In particular, when $P(f) = f^n$, let $H(z) = \frac{f(z)}{g(z)}$. Next proceeding in the same manner when $b = -1$, in Case 2, we can show that $H(z)$ must be a constant and $f \equiv tg$, where $t^{n+1} = 1$. This proves the theorem. □

Proof of Theorem 1.2. In both cases $H \not\equiv 0$ and $H \equiv 0$, we put $\Gamma_0 = m+1$, $\Gamma_1 = 2(m+1)$ and proceed in the same manner as done in *Theorem 1.1* to get

$$f^r(z)(f^m(z)-1)^p f(qz) \equiv g^r(z)(g^m(z)-1)^p g(qz)$$

or

$$f^r(z)(f^m(z)-1)^p f(qz).g^r(z)(g^m(z)-1)^p g(qz) \equiv 1. \tag{4.11}$$

Next we adopt the idea of Theorem 3.4 in [16] and prove that $f^r(z)(f^m(z)-1)^p f(qz).g^r(z)(g^m(z)-1)^p g(qz) \equiv 1$ does not occur. Let $h(z) = f(z)g(z)$, then rewriting (4.11) we have

$$h(z)^r(h(z)^m - (f(z)^m + g(z)^m) + 1)^p h(qz) = 1. \tag{4.12}$$

Case 1. Suppose $h(z)$ is not a constant. Suppose that there exists a point z_0 such that $h(z_0) = 0$, which implies $f(z_0)g(z_0) = 0$. Since f and g share $(\infty,0)$, we get $f(z_0) \neq \infty$ and $g(z_0) \neq \infty$. Then by (4.12), we conclude that $h(qz_0) = \infty$. So,

$$h(z) = 0 \implies h(qz) = \infty. \tag{4.13}$$

Now suppose that there is a point z_1 such that $h(qz_1) = 0$, from (4.12), we have $h(z_1) = \infty$, or else, if $h(z_1) \neq \infty$, then $f(z_1) \neq \infty$ and $g(z_1) \neq \infty$, from which we get a contradiction by (4.12). Hence

$$h(qz) = 0 \implies h(z) = \infty. \tag{4.14}$$

Next assume that there is a point z_2 such that $h(z_2) = \infty$ and z_2 is a pole of f with multiplicity s and a pole of g with multiplicity t. Then z_2 is a pole of $h(z)^r$ with multiplicity $r(s+t) < (r+m)(s+t)$, a pole of $h(z)^m$ with multiplicity $m(s+t) < (r+m)(s+t)$, a pole of $f(z)^m + g(z)^m$ with multiplicity $m\max(s,t) < (r+m)(s+t)$. So, z_2 is a pole of $h(z)^r(h(z)^m - (f(z)^m + g(z)^m) + 1)^p$ with multiplicity at most $p(r+m)^2(s+t)^2$. Hence $h(qz_2) = 0$, which implies that

$$h(z) = \infty \implies h(qz) = 0. \tag{4.15}$$

If possible let $h(qz)=\infty \implies h(z)\neq 0$, then from (4.13) and (4.15),

$$h(qz)=\infty \implies h(q^2z)=0 \implies h(q^3z)=\infty \implies h(q^2z)\neq 0,$$

which is impossible. Therefore,

$$h(qz)=\infty \implies h(z)=0. \tag{4.16}$$

If $|q|<1$, then from (4.13) and (4.15) we have

$$\begin{aligned} &h(z)=0 \implies h(qz)=\infty \implies h(q^2z)=0 \implies \cdots \\ &\implies h(q^{2k}z)=0 \implies h(q^{2k+1}z)=\infty\cdots, \end{aligned}$$

where k is a positive integer. So, we get

$$0=\lim_{z\to 0} h(z)=\infty,$$

a contradiction.
If $|q|>1$, then from (4.14) and (4.16) we get

$$\begin{aligned} &h(qz)=0 \implies h(z)=\infty \implies h\left(\frac{z}{q}\right)=0 \implies \cdots \\ &\implies h\left(\frac{z}{q^{2k}}\right)=\infty \implies h\left(\frac{z}{q^{2k+1}}\right)=0\cdots, \end{aligned}$$

where k is a positive integer. So in a similar way we get a contradiction.
Case 2. Let $h(z)$ is non-zero constant, say t, i.e., $f(z)g(z)=t$. Since f and g share $(\infty,0)$, we can easily get $f(z)$ and $g(z)$ have no zeros and no poles. That means the orders of f and g are not less than 1 but we assumed that $f(z)$ and $g(z)$ are of zero order.
Hence, $f^r(z)(f^m(z)-1)^p f(qz).g^r(z)(g^m(z)-1)^p g(qz)\equiv 1$ is not possible, which means $f^r(z)(f^m(z)-1)^p f(qz)\equiv g^r(z)(g^m(z)-1)^p g(qz)$. This completes the proof of the theorem. □

Proof of Theorem 1.3. Here we put $p=1$, $\Gamma_0=m+1$ and $\Gamma_1=m+2$ and proceed similarly as in Theorem 1.2, we have the conclusion. □

Proof of Theorem 1.4. We follow the method of Theorem 1.5 in [21] and prove the theorem in the following manner. Let $\phi=(F(z))^{(k)}=(P(f)(z)f(qz+c))^{(k)}$ and $\psi=(G(z))^{(k)}=(P(g)(z)g(qz+c))^{(k)}$. Then ϕ and ψ share $(1,l)$. Applying *Lemmas 3.1*, *3.4* and *3.12*, we have

$$\begin{aligned} N_2(r,0;\phi)&=N_2(r,0;F^{(k)}) \\ &\leq k\overline{N}(r,\infty;F)+N_{k+2}(r,0;F)+S(r,f) \\ &\leq \overline{N}(r,\infty;P(f))+k\overline{N}(r,\infty;f(qz+c)) \\ &\quad +N_{k+2}(r,0;P(f))+N_{k+2}(r,0;f(qz+c))+S(r,f) \end{aligned}$$

$$\begin{aligned}
&\leq 2kT(r,f)+(m_1+(k+2)m_2)T(r,f)+T(r,f)+S(r,f)\\
&\leq 2kT(r,f)+(m_1+(k+2)m_2+1)T(r,f)+S(r,f)\\
&\leq ((m_2+2)k+\Gamma_1+1)T(r,f)+S(r,f), \qquad (4.17)
\end{aligned}$$

$$\begin{aligned}
N_2(r,\infty;\phi) &= N_2(r,\infty;F^{(k)})+S(r,f)\\
&\leq N_2(r,\infty;F)+S(r,f)\\
&\leq N_2(r,\infty;P(f))+N_2(r,\infty;f(qz+c))+S(r,f)\\
&\leq 2T(r,f)+T(r,f)+S(r,f)=3T(r,f)+S(r,f), \qquad (4.18)
\end{aligned}$$

$$\begin{aligned}
\overline{N}(r,0;\phi) &= \overline{N}(r,0;F^{(k)})+S(r,f)\\
&\leq k\overline{N}(r,\infty;F)+N_{k+1}(r,0;F)+S(r,f)\\
&\leq k\overline{N}(r,\infty;P(f))+k\overline{N}(r,\infty;f(qz+c))\\
&\quad +N_{k+1}(r,0;P(f))+N_{k+1}(r,0;f(qz+c))+S(r,f)\\
&\leq 2kT(r,f)+(m_1+(k+1)m_2)T(r,f)+T(r,f)+S(r,f)\\
&\leq 2kT(r,f)+(m_1+(k+1)m_2+1)T(r,f)+S(r,f)\\
&\leq ((m_2+2)k+\Gamma_0+1)T(r,f)+S(r,f) \qquad (4.19)
\end{aligned}$$

and

$$\begin{aligned}
\overline{N}(r,\infty;\phi) &= \overline{N}(r,\infty;F^{(k)})+S(r,f)\\
&\leq \overline{N}(r,\infty;F)+S(r,f)\\
&\leq \overline{N}(r,\infty;P(f))+\overline{N}(r,\infty;f(qz+c))+S(r,f)\\
&\leq 2T(r,f)+S(r,f). \qquad (4.20)
\end{aligned}$$

Here two cases arise.

Case-1. Let $H \not\equiv 0$.
Now, applying *Lemma 3.12*, we have

$$\begin{aligned}
N_2(r,0;\phi) &\leq N_2(r,0;F^{(k)})+S(r,f)\\
&\leq T(r,F^{(k)})-T(r,F)+N_{k+2}(r,0;F)+S(r,f)\\
&\leq T(r,\phi)-T(r,F)+N_{k+2}(r,0;F)+S(r,f)
\end{aligned}$$

i.e.,

$$T(r,F)\leq T(r,\phi)-N_2(r,0;\phi)+N_{k+2}(r,0;F)+S(r,f). \qquad (4.21)$$

Combining *Lemma 3.13* and (4.21), we have

$$\begin{aligned}
(n-1)T(r,f) &\leq T(r,F)\\
&\leq T(r,\phi)-N_2(r,0;\phi)+N_{k+2}(r,0;F)+S(r,f). \qquad (4.22)
\end{aligned}$$

Subcase 1.1. While $l \geq 2$, in view of case (i) of *Lemma 3.9*, using (4.17), (4.18) and (4.22) we have

$$\begin{aligned}
(n-1)T(r,f) \leq\; & N_2(r,0;\psi)+N_2(r,\infty;\phi)+N_2(r,\infty;\psi)+N_{k+2}(r,0;F)\\
&+S(r,f)+S(r,g)
\end{aligned}$$

$$\begin{aligned}&\le ((m_2+2)k+\Gamma_1+1)T(r,g)+(km_2+\Gamma_1+1)T(r,f)\\&\quad+3T(r,f)+3T(r,g)+S(r,f)+S(r,g)\\&\le (km_2+\Gamma_1+4)T(r,f)+((m_2+2)k+\Gamma_1+4)T(r,g)\\&\quad+S(r,f)+S(r,g).\end{aligned}$$

Similarly,

$$\begin{aligned}(n-1)T(r,g)\le{}& ((m_2+2)k+\Gamma_1+4)T(r,f)+(km_2+\Gamma_1+4)T(r,g)\\&+S(r,f)+S(r,g).\end{aligned}$$

Combining the above two equations, we have

$$\begin{aligned}(n-1)[T(r,f)+T(r,g)]\le{}& (2(m_2+1)k+2\Gamma_1+8)[T(r,f)+T(r,g)]\\&+S(r,f)+S(r,g),\end{aligned}$$

which is a contradiction for $n>2(m_2+1)k+2\Gamma_1+9$.

Subcase 1.2. While $l=1$, in view of *Lemma 3.10*, using (4.17), (4.18), (4.19), (4.20) and (4.22), we have

$$\begin{aligned}&(n-1)T(r,f)\\&\quad\le N_2(r,0;\psi)+N_2(r,\infty;\phi)+N_2(r,\infty;\psi)+N_{k+2}(r,0;F)+\frac{1}{2}\overline{N}(r,0;\phi)\\&\qquad+\frac{1}{2}\overline{N}(r,\infty;\phi)+S(r,f)+S(r,g)\\&\quad\le ((m_2+2)k+\Gamma_1+1)T(r,g)+(km_2+\Gamma_1+1)T(r,f)+3T(r,f)\\&\qquad+3T(r,g)+\frac{1}{2}((m_2+2)k+\Gamma_0+1)T(r,f)+T(r,f)+S(r,f)+S(r,g)\\&\quad\le\left[\left(\frac{3}{2}m_2+1\right)k+\Gamma_1+\frac{1}{2}\Gamma_0+\frac{11}{2}\right]T(r,f)\\&\qquad+[(m_2+2)k+\Gamma_1+4]T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

Similarly,

$$\begin{aligned}(n-1)T(r,g)\le{}& ((m_2+2)k+\Gamma_1+4)T(r,f)\\&+\left[\left(\frac{3}{2}m_2+1\right)k+\Gamma_1+\frac{1}{2}\Gamma_0+\frac{11}{2}\right]T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

Combining the above two equations, we have

$$\begin{aligned}&(n-1)[T(r,f)+T(r,g)]\\&\quad\le\left[\left(\frac{5}{2}m_2+3\right)k+2\Gamma_1+\frac{1}{2}\Gamma_0+\frac{19}{2}\right][T(r,f)+T(r,g)]+S(r,f)+S(r,g),\end{aligned}$$

which is a contradiction for $n>\left(\frac{5}{2}m_2+3\right)k+2\Gamma_1+\frac{1}{2}\Gamma_0+\frac{21}{2}$.

Subcase 1.3. While $l=0$, in view of *Lemma 3.11*, using (4.17), (4.18), (4.19), (4.20) and (4.22) we have

$$\begin{aligned}
&(n-1)T(r,f)\\
&\quad\le N_2(r,0;\psi)+N_2(r,\infty;\phi)+N_2(r,\infty;\psi)+N_{k+2}(r,0;F)\\
&\qquad+2\overline{N}(r,0;\phi)+2\overline{N}(r,\infty;\phi)+\overline{N}(r,0;\psi)+\overline{N}(r,\infty;\psi)+S(r,f)+S(r,g)\\
&\quad\le((m_2+2)k+\Gamma_1+1)T(r,g)+(km_2+\Gamma_1+1)T(r,f)\\
&\qquad+3T(r,f)+3T(r,g)+2((m_2+2)k+\Gamma_0+1)T(r,f)\\
&\qquad+((m_+2)k+\Gamma_0+1)T(r,g)+4T(r,f)+2T(r,g)+S(r,f)+S(r,g)\\
&\quad\le((3m_2+4)k+2\Gamma_0+\Gamma_1+10)\,T(r,f)\\
&\qquad+(2(m_2+2)k+\Gamma_1+\Gamma_0+7)T(r,g)+S(r,f)+S(r,g).
\end{aligned}$$

Similarly,

$$\begin{aligned}
(n-1)T(r,g)\le\ &(2(m_2+2)k+\Gamma_1+\Gamma_0+7)T(r,f)\\
&+((3m_2+4)k+2\Gamma_0+\Gamma_1+10)\,T(r,g)+S(r,f)+S(r,g).
\end{aligned}$$

Combining the above two equations, we have

$$\begin{aligned}
&(n-1)[T(r,f)+T(r,g)]\\
&\quad\le((5m_2+8)k+2\Gamma_1+3\Gamma_0+17)[T(r,f)+T(r,g)]+S(r,f)+S(r,g),
\end{aligned}$$

which is a contradiction for $n>(5m_2+8)k+2\Gamma_1+3\Gamma_0+18$.

Case-2. Let $H\equiv 0$. By integration, we get

$$\frac{1}{\phi-1}\equiv\frac{b\psi+a-b}{\psi-1}, \tag{4.23}$$

where $a(\neq 0)$, b are constants. From (4.23), it is clear that ϕ and ψ share $(1,\infty)$. We consider the following cases:

Subcase 2.1. Let $b\neq 0$ and $a\neq b$. If $b=-1$, then from (4.23) we have

$$\phi\equiv\frac{-a}{\psi-a-1}.$$

From *Lemma 3.4* and (4.20), we see that

$$\overline{N}(r,a+1;\psi)=\overline{N}(r,\infty;\phi)\le 2\overline{N}(r,\infty;f).$$

So, using the second fundamental theorem, we get

$$\begin{aligned}
T(r,\psi)&\le\overline{N}(r,0;\psi)+\overline{N}(r,\infty;\psi)+\overline{N}(r,a+1;\psi)+S(r,g)\\
&\le\overline{N}(r,0;\psi)+\overline{N}(r,\infty;\psi)+\overline{N}(r,\infty;\phi)+S(r,f)+S(r,g).
\end{aligned}$$

By *Lemma 3.12*, we see

$$\overline{N}(r,0;\psi)\le T(r,\psi)-T(r,G)+N_{k+1}(r,0;G)+S(r,g).$$

These two inequalities imply

$$T(r,G)\le\overline{N}(r,\infty;\psi)+\overline{N}(r,\infty;\phi)+N_{k+1}(r,0;G)+S(r,f)+S(r,g).$$

From the above equation, using (4.20) and *Lemmas 3.4*, *3.13*, we have

$$\begin{aligned}(n-1)T(r,g) &\leq \overline{N}(r,\infty;\psi)+\overline{N}(r,\infty;\phi)+N_{k+1}(r,0;G)+S(r,f)+S(r,g)\\ &\leq 2T(r,f)+2T(r,g)+(m_1+(k+1)m_2+1)T(r,g)+S(r,f)+S(r,g)\\ &\leq 2T(r,f)+(\Gamma_1+km_2+3)T(r,g)+S(r,f)+S(r,g).\end{aligned}$$

As ϕ and ψ are interchangeable, in a similar manner we can get

$$(n-1)\ T(r,f)\leq(\Gamma_1+km_2+3)T(r,f)+2T(r,g)+S(r,f).$$

Combining the above two, we can get

$$(n-1)\{T(r,f)+T(r,g)\}\leq(\Gamma_1+km_2+5)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),$$

a contradiction for $n>2(m_2+1)k+2\Gamma_1+9$.
If $b\neq-1$, from (4.23) we obtain that

$$\phi-\left(1+\frac{1}{b}\right)\equiv\frac{-a}{b^2[\psi+\frac{a-b}{b}]}.$$

So,

$$\overline{N}\left(r,\frac{(b-a)}{b};\psi\right)=\overline{N}(r,\infty;\phi).$$

Using *Lemmas 3.4*, *3.12*, *3.13* and with the same argument as used in the case for $b=-1$, we can get a contradiction.

Subcase 2.2. Let $b\neq0$ and $a=b$. If $b=-1$, then from (4.23) we have

$$\phi\psi\equiv1,$$

i.e.,

$$(P(f)(z)f(qz+c))^{(k)}(P(g)(z)g(qz+c))^{(k)}\equiv1.$$

If $b\neq-1$, from (4.23) we have

$$\frac{1}{\phi}\equiv\frac{b\psi}{(1+b)\psi-1}.$$

Therefore,

$$\overline{N}\left(r,\frac{1}{1+b};\psi\right)=\overline{N}(r,0;\phi).$$

So, using the second fundamental theorem, we get

$$\begin{aligned}T(r,\psi) &\leq \overline{N}(r,0;\psi)+\overline{N}(r,\infty;\psi)+\overline{N}\left(r,\frac{1}{1+b};\psi\right)+S(r,g)\\ &\leq \overline{N}(r,0;\psi)+\overline{N}(r,\infty;\psi)+\overline{N}(r,0;\phi)+S(r,f)+S(r,g).\end{aligned}$$

By *Lemma 3.12*, we see

$$\overline{N}(r,0;\psi)\leq T(r,\psi)-T(r,G)+N_{k+1}(r,0;G)+S(r,g).$$

These two equations imply

$$T(r,G)\leq\overline{N}(r,\infty;\psi)+\overline{N}(r,0;\phi)+N_{k+1}(r,0;G)+S(r,f)+S(r,g).$$

From the above equation, using (4.19), (4.20) and *Lemmas 3.4*, *3.13*, we have

$$\begin{aligned}
&(n-1)T(r,g)\\
&\quad\le \overline{N}(r,\infty;\psi)+\overline{N}(r,0;\phi)+N_{k+1}(r,0;G)+S(r,f)+S(r,g)\\
&\quad\le ((m_2+2)k+\Gamma_0+1)T(r,f)+2T(r,g)\\
&\qquad+(km_2+\Gamma_1+1)T(r,g)+S(r,f)+S(r,g)\\
&\quad\le ((m_2+2)k+\Gamma_0+1)T(r,f)+(km_2+\Gamma_1+3)T(r,g)+S(r,f)+S(r,g).
\end{aligned}$$

As ϕ and ψ are symmetric, in a similar manner, we can get

$$(n-1)\ T(r,f)\le (km_2+\Gamma_1+3)T(r,f)+((m_2+2)k+\Gamma_0+1)T(r,g)+S(r,f).$$

Combining the above two, we can get

$$\begin{aligned}
&(n-1)\{T(r,f)+T(r,g)\}\\
&\qquad\le (2(m_2+1)k+\Gamma_1+\Gamma_0+4)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),
\end{aligned}$$

a contradiction for $n>2(m_2+1)k+2\Gamma_1+9$.

Subcase 2.3. Let $b=0$. From (4.23), we obtain

$$\phi\equiv\frac{\psi+a-1}{a}. \tag{4.24}$$

If $a\ne 1$, then from (4.24), we obtain

$$\overline{N}(r,1-a;\psi)=\overline{N}(r,0;\phi).$$

So, using the same argument as done in Case 2, for $b\ne -1$, we can similarly deduce a contradiction. Therefore $a=1$ and from (4.24) we obtain $\phi\equiv\psi$, i.e.,

$$(P(f)(z)f(qz+c))^{(k)}\equiv (P(g)(z)g(qz+c))^{(k)}.$$

Integrating we have $P(f)(z)f(qz+c)=P(g)(z)g(qz+c)+p(z)$, where $p(z)$ is a polynomial of degree at most $k-1$.

If $p(z)\not\equiv 0$, then from the second main theorem for the small function and *Lemma 3.13*, we get

$$\begin{aligned}
(n-1)T(r,f)&\le T(r,F)+S(r,f)\\
&\le \overline{N}(r,F)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)\\
&\le (\Gamma_0+3)T(r,f)+(\Gamma_0+1)T(r,g)+S(r,f).
\end{aligned}$$

Similarly,

$$(n-1)T(r,g)\le (\Gamma_0+3)T(r,g)+(\Gamma_0+1)T(r,f)+S(r,g).$$

Therefore,

$$(n-1)[T(r,f)+T(r,g)]\le (2\Gamma_0+4)[T(r,f)+T(r,g)]+S(r,f)+S(r,g),$$

which by $n>2(m_2+1)k+2\Gamma_1+9$ gives a contradiction. Thus $p(z)\equiv 0$, which implies

$$P(f)(z)f(qz+c)=P(g)(z)g(qz+c). \tag{4.25}$$

Let $h(z)=f(z)g(z)$. Then the following two cases hold.

Case A. Suppose that $h(z) \equiv$ constant, say h. Substituting $f(z) = hg(z)$ into (4.25), we obtain

$$g(qz+c)[a_n g(z)^n (h^{n+1}-1) + a_{n-1} g(z)^{n-1} (h^n - 1) + \cdots + a_1 g(z)(h^2-1) + a_0(h-1)] \equiv 0.$$

Since $g(z)$ is a non-constant meromorphic function, we have $g(qz+c) \not\equiv 0$. Hence, we get

$$a_n g(z)^n (h^{n+1}-1) + a_{n-1} g(z)^{n-1} (h^n - 1) + \cdots + a_1 g(z)(h^2-1) + a_0(h-1) \equiv 0. \tag{4.26}$$

We shall prove that $h^\lambda = 1$, where λ is the GCD of the elements of J, $J = \{k+1 \in I : a_k \neq 0\}$ and $I = \{1,2,\ldots,n+1\}$. In particular, if $P(z) = a_n z^n$, then from above we get $h^{n+1} = 1$. Thus $f \equiv tg$ for a constant t such that $t^{n+1} = 1$. Suppose there exists at least one non-zero coefficient a_k, $k \neq n$. Then if $h^\lambda \neq 1$, from (4.26) we get $T(r,g) = S(r,g)$, a contradiction to the fact that g is transcendental. So $h^\lambda = 1$, where λ is the GCD of the elements of J, $J = \{k+1 \in I : a_k \neq 0\}$ and $I = \{1,2,\ldots,n+1\}$.

Case B. Suppose that $h(z)$ is not a constant. we deduce from (4.25) that $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z), g(z)) = 0$, where
$R(w_1, w_2) = P(w_1)w_1(qz+c) - P(w_2)w_2(qz+c)$. This completes the proof. □

Proof of Corollary 1.3. The corollary can be proved in the line of the proof of *Theorem 1.4* with necessary changes. For example, one has to use *Lemma 3.14* instead of *Lemma 3.13*. So we omit the details. □

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