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COMBINATORIAL INTERPRETATIONS OF PRIMITIVITY IN THE ALGEBRA OF SYMMETRIC FUNCTIONS

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ABSTRACT. Given a Hopf algebra with distinguished bases indexed by combinatorial objects, along with a primitive generating set for this algebra, it is natural to consider how we can combinatorially interpret the primitivity of the elements in this generating set, by expanding these generators in terms of the distinguished bases of this algebra and then applying the comultiplication operation to these expansions and constructing sign-reversing involutions that determine the resultant cancellations that we obtain. In this article, we explore this idea, as applied to the power sum generators of the algebra of symmetric functions.

1. INTRODUCTION

The algebra Sym of symmetric functions plays an extremely important role in many areas in mathematics, especially within the realms of representation theory and algebraic combinatorics, not to mention the uses of symmetric functions in algebraic geometry, group theory, and the study of Lie algebras [10, p. 286]. Letting $\mathbb{k}$ be a field of characteristic 0, we may let Sym be defined as the free commutative $\mathbb{k}$-algebra with one generator in each degree; one might anticipate that such a simple definition would provide a structure that would naturally arise in many different areas of research in mathematics and physics. Inspired by the sign-reversing involutions introduced in [2] applied to Takeuchi's formula to determine cancellation-free formulas for the antipodes of combinatorial Hopf algebras, as well as the sign-reversing involutions introduced in [3] to prove new coproduct formulas for noncommutative analogues of the Schur basis of Sym, we are interested in applying similar kinds of bijective approaches so as to obtain new combinatorial results concerning primitive elements in the Hopf algebra structure with which we endow Sym.

Writing $\mathsf{Sym}_n = \mathbb{k}[x_1, x_2, \ldots, x_n]^{S_n}$, recall that the Fundamental Theorem of Symmetric Polynomials gives us that Sym_n forms a ring that is generated by the elementary symmetric polynomials $e_k = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ for $k \leq n$, i.e., so that

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we may write

$$\mathsf{Sym}_n = \Bbbk[e_1, e_2, \ldots, e_n]. \tag{1.1}$$

With regard to the definition of the algebra Sym that we had noted above, we let $n \to \infty$ in (1.1) (cf. [8]), writing $\mathsf{Sym} = \Bbbk[e_1, e_2, \ldots]$. Recall that the power sum symmetric polynomial of order $k \in \mathbb{N}$ may be defined so that $p_k = \sum_{i=1}^{n} x_i^k$ for a given number of $n \in \mathbb{N}$ variables, with

$$\mathsf{Sym}_n = \Bbbk[p_1, p_2, \ldots, p_n].$$

As above, we let $n \to \infty$, giving us the *power sum generators* of Sym, as below:

$$\mathsf{Sym} = \Bbbk[p_1, p_2, \ldots]. \tag{1.2}$$

For a bialgebra A with a coproduct $\Delta \colon A \to A \otimes A$, recall that a primitive element in such an algebra is an element $x \in A$ satisfying the property that $\Delta(x) = 1 \otimes x + x \otimes 1$. A fundamental property concerning the generating set suggested in (1.2) is given by each power sum generator being primitive. Since the notion of primitivity plays an important role in the theory of symmetric functions and in many other areas in algebraic combinatorics, this motivates our considering how we can make sense of the primitivity of the members of the generating set $\{p_n\}_{n \in \mathbb{N}}$ in a combinatorial way, using sign-reversing involutions.

Recall that bases of Sym are indexed by integer partitions, i.e., letting a *partition* of an integer $m \in \mathbb{N}$ refer to a tuple of weakly decreasing natural numbers such that the sum of the entries of this tuple equals m; we generalize this definition by defining a *composition* of a natural number m as an arbitrary tuple of natural numbers such that the entries of this tuple sum to m. Given a composition α, we let $\ell(\alpha)$ denote the *length* of this tuple, i.e., the number of entries in this tuple, and we write $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)})$. We also write $\lambda \vdash m$ to indicate that λ is a partition of m, and we define the *order* $|\alpha|$ of a composition α as the sum of the entries of α.

We also extend our definition of the term "integer partition" so that $() \vdash 0$, i.e., so that the empty tuple $()$ is the unique partition of $0 \in \mathbb{N}_0 = \{0, 1, \ldots\}$, and we write $e_{()} = e_0 = p_{()} = p_0 = 1$. Since integer partitions are of such importance in combinatorics, and since a given basis of Sym is indexed by the set of these combinatorial objects, it is natural that the distinguished bases of Sym would closely reflect combinatorial properties associated with integer partitions, and similar kinds of discrete objects. An aspect about the study of symmetric functions that is of essential importance in this field of study comes from the identities that we obtain for evaluating the entries of the transition matrices between the distinguished bases of Sym [1], and this forms a central idea in this article, the main purpose of which is to explore the following problem: If we expand the generator p_k in terms of one of the usual bases of Sym, and then apply Δ to each term in this expansion, and then expand this resultant expression as a linear combination of simple tensors, then how can we construct a bijection that determines any cancellations in this latter expansion, in some kind of combinatorial way that reflects the distinguished basis of Sym under consideration?

In general, expanding coproducts of elements in Sym can be complicated, in contrast to how fundamentally simple the comultiplication formula $\Delta(p_k) = 1 \otimes p_k + p_k \otimes 1$ is. Since the right-hand side of this equality is just a sum of two simple tensors, we may anticipate that there would, in general, be a very large amount of cancellation that may occur after the expansion processes indicated in the above question in the preceding paragraph. However, it may be reasonable to suggest that this kind of cancellation would occur in a very "well-behaved" way, and, as suggested above, in a way that is closely associated with the combinatorics behind the evaluation of the transition matrices between the classical bases of Sym. Similar kinds of ideas have been explored in the references [2] and [3], and also in [6] and [7].

For an integer partition λ, we write $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{\ell(\lambda)}}$. The set of all such expressions, i.e., the family $\{p_\lambda\}_{\lambda \in \mathcal{P}}$, is, of course, one of the distinguished bases of Sym, namely, the *power sum basis* of this algebra, letting $\mathcal{P}$ denote the set of all integer partitions. Also, writing $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_{\ell(\lambda)}}$, the set $\{e_\lambda\}_{\lambda \in \mathcal{P}}$ is the *elementary basis* of Sym. In our exploring the main problem that our article is based upon, as described above, we begin with the expansion of p_n into the e-basis.

2. A BIJECTION ON COLOURED COMPOSITION TABLEAUX

We may expand a given power sum generator using elementary generators according to the following determinantal identity [11]:

$$p_n = \begin{vmatrix} ne_n & e_{n-1} & e_{n-2} & \cdots & e_1 \\ (n-1)e_{n-1} & e_{n-2} & e_{n-3} & \cdots & 1 \\ (n-2)e_{n-2} & e_{n-3} & e_{n-4} & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ e_1 & 1 & 0 & \cdots & 0 \end{vmatrix}.$$

In our offering an explicit, cancellation-free formula for evaluating power sum generators in terms of $\{e_\lambda\}_{\lambda \in \mathcal{P}}$, we make use of the notation given as follows. For partitions λ and μ, let $\lambda \uplus \mu$ denote the partition obtained by sorting the concatenation of the entries of λ and μ. Let us also write $\lambda = (k^{m_k(\lambda)}, \ldots, 2^{m_2(\lambda)}, 1^{m_1(\lambda)})$. In general, we have that

$$p_\mu = (-1)^{|\mu| - \ell(\mu)} \sum_{\lambda \vdash |\mu|} E_{\lambda\mu} e_\lambda,$$

where

$$E_{\lambda\mu} = (-1)^{\ell(\lambda)-\ell(\mu)} \sum_{\left(\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(\ell(\mu))}\right)} \prod_{i=1}^{\ell(\mu)} \frac{\mu_i (\ell(\nu^{(i)}) - 1)!}{\prod_{j \geq 1} m_j(\nu^{(i)})!}, \tag{2.1}$$

and where the above sum is over all tuples of partitions with $\nu^{(i)} \vdash \mu_i$ and $\nu^{(1)} \uplus \nu^{(2)} \uplus \cdots \uplus \nu^{(\ell(\mu))} = \lambda$. So, we have that

$$p_n = (-1)^{n-1} \sum_{\lambda \vdash n} E_{\lambda,n} e_\lambda, \tag{2.2}$$

where $E_{\lambda,n} = (-1)^{\ell(\lambda)-1} \frac{n(\ell(\lambda)-1)!}{\prod_{j\geq 1} m_j(\lambda)!}$, adopting notation from [11]. For example, we obtain the expansion whereby:

$$\begin{aligned} p_{(6)} &= -6e_{(6)} + 6e_{(5,1)} + 6e_{(4,2)} - 6e_{(4,1,1)} + 3e_{(3,3)} - 12e_{(3,2,1)} \\ &+ 6e_{(3,1,1,1)} - 2e_{(2,2,2)} + 9e_{(2,2,1,1)} - 6e_{(2,1,1,1,1)} + e_{(1,1,1,1,1,1)}. \end{aligned}$$

The coefficients that we obtain in the expansion of power sum generators into the elementary basis of Sym are also the coefficients of what are referred to as *Faber partition polynomials*, as indexed in the entry A263916 in The On-Line Encyclopedia of Integer Sequences [9]. Our goal is to find some kind of combinatorial interpretation of the primitivity of p_n by writing

$$\Delta(p_n) = \sum_{\lambda \vdash n} (-1)^{n+\ell(\lambda)} \frac{n(\ell(\lambda)-1)!}{\prod_{j\geq 1} m_j(\lambda)!} \Delta(e_\lambda) \tag{2.3}$$

and expanding the coproducts in the above sum, and then using a bijection to show that we do indeed obtain the primitivity of p_n. So, what does the expression

$$\frac{n(\ell(\lambda)-1)!}{\prod_{j\geq 1} m_j(\lambda)!} \tag{2.4}$$

"count", and what kind of combinatorial interpretation of these coefficients could we apply to obtain such a bijective proof?

With regard to the given denominator on the right-hand side of (2.3), since the sum $m_1(\lambda) + m_2(\lambda) + \cdots$ gives us the length of the partition λ, it would make sense to make use of *multinomial coefficients*, in consideration of what is discussed below. Multinomial coefficients are expressions of the form

$$(n_1, n_2, \ldots, n_k)! = \frac{(n_1 + n_2 + \cdots + n_k)!}{n_1! n_2! \cdots n_k!},$$

and the above expression gives us the number of distinct permutations in a multiset of k different members of multiplicity n_i. Let us rewrite the coefficient displayed in (2.4) as below:

$$\frac{\lambda_1 + \lambda_2 + \cdots + \lambda_{\ell(\lambda)}}{\ell(\lambda)} \cdot \frac{\ell(\lambda)!}{\prod_{j\geq 1} m_j(\lambda)!}. \tag{2.5}$$

We can see that the multinomial coefficient $\frac{\ell(\lambda)!}{\prod_{j\geq 1} m_j(\lambda)!}$ is equal to the number of compositions that have the same length as λ and the same entries as λ. So, if we think of all of the cells in the diagram of λ as being distinct, we can think of the product

$$|\lambda| \cdot \frac{\ell(\lambda)!}{\prod_{j\geq 1} m_j(\lambda)!} \tag{2.6}$$

as being the cardinality of the Cartesian product of the set of these cells and the compositions with the same entries and length as λ.

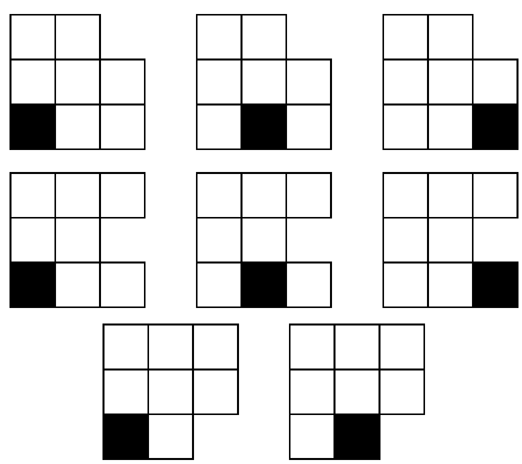
We assume familiarity with Ferrers diagram, partition diagrams, composition tableaux, etc., and we note that we make use of what is referred to as "French notation" for denoting such diagrams. Returning our attention to the value in (2.6), a natural interpretation of this expression would give us that (2.6) equals the set of all distinct expressions of the following form: Compositions with the same entries and length as λ, written as diagrams and with exactly one cell colored.

By dividing (2.6) by $\ell(\lambda)$, we think of dividing the set of tableaux defined above that we associate with (2.6) into $\ell(\lambda)$ sets of equal cardinality. So, it is natural to let these $\ell(\lambda)$ different subsets be indexed by the rows of the diagram of λ, so that each of these sets somehow "corresponds" to one of the rows of λ. We restrict our attention to colored tableaux corresponding to (2.6) in which the unique colored cell is in a fixed row, say, the initial row.

Example 2.1. *Let us consider the expansion*

$$\begin{aligned} p_8 &= e_{(1^8)} - 8e_{(2,1^6)} + 20e_{(2^2,1^4)} - 16e_{(2^3,1^2)} + 2e_{(2^4)} + 8e_{(3,1^5)} - 32e_{(3,2,1^3)} \\ &+ 24e_{(3,2^2,1)} + 12e_{(3^2,1^2)} - 8e_{(3^2,2)} - 8e_{(4,1^4)} + 24e_{(4,2,1^2)} - 8e_{(4,2^2)} \\ &- 16e_{(4,3,1)} + 4e_{(4^2)} + 8e_{(5,1^3)} - 16e_{(5,2,1)} + 8e_{(5,3)} - 8e_{(6,1^2)} + 8e_{(6,2)} \\ &+ 8e_{(7,1)} - 8e_{(8)}, \end{aligned}$$

and let us focus our attention on the above coefficient for the basis element $e_{(3^2,2)} \in \mathsf{Sym}$*, namely:* -8. *Adopting notation from* (2.3)*, we have that* $n = 8$ *and* $\lambda = (3,3,2)$*, so the sign of this coefficient is* $(-1)^{n+\ell(\lambda)} = -1$*, as expected. According to our combinatorial interpretation for the coefficients in the expansion of* p_n *into* $\{e_\mu\}_{\mu \in \mathcal{P}}$*, we should have that there are exactly* 8 *compositions with the same length and entries as* λ*, and with one colored cell in the initial row. As indicated below, this turns out to be correct.*



Now that we have determined a suitable combinatorial interpretation of the absolute values of the scalars in the summand in (2.3), we now intend to expand the coproduct within this summand, and then create yet another family of combinatorial objects to associate with the absolute values of the coefficients that we then obtain. We may take the coproduct identity

$$\Delta(e_n) = e_{()} \otimes e_n + e_1 \otimes e_{n-1} + \cdots + e_{n-1} \otimes e_1 + e_n \otimes e_{()} \tag{2.7}$$

as a definition, in our proving the primitivity of power sum generators combinatorially.

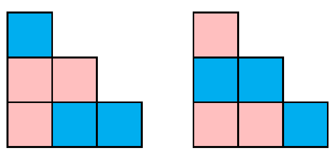
Through repeated applications of the identity in (2.7), we obtain a natural combinatorial interpretation for the coefficients that we obtain in the expansion of coproducts of elements in the elementary basis [11]. For a partition λ, let us write

$$\Delta(e_\lambda) = \sum_{\mu,\nu} C_{\mu,\nu}\, e_\mu \otimes e_\nu, \tag{2.8}$$

where we may take the above sum to be over all partitions μ and ν such that $|\lambda| = |\mu| + |\nu|$. As noted in [11], although in reference to the complete homogeneous basis $\{h_\lambda\}_{\lambda\in\mathcal{P}}$, for which the same combinatorial rule still holds, $C_{\mu,\nu}$ counts the number of colorings of λ whereby:

(1) We choose to fix two colors, say, pink and cyan, so that each row of λ is colored with a possibly zero number of consecutive pink cells on the left and a possibly zero number of consecutive cyan cells on the right; and
(2) If we take the pink (resp. cyan) cells in each row, this gives us a composition with the same entries as μ, or ν, respectively.

Example 2.2. *For example, taking the coproduct $\Delta(e_{(3,2,1)})$, we see that the coefficient of $e_{(2,1)} \otimes e_{(2,1)}$ is 2. This agrees with the combinatorial interpretation indicated above, as suggested below.*



So, we obtain the equality

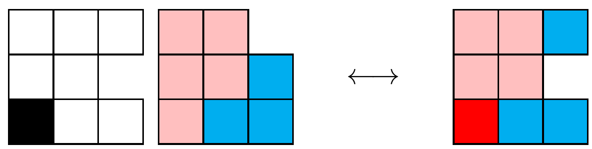
$$\Delta(p_n) = (-1)^n \sum_{\lambda\vdash n} \sum_{|\mu|+|\nu|=n} (-1)^{\ell(\lambda)} \frac{n(\ell(\lambda)-1)!}{\prod_{j\geq 1} m_j(\lambda)!} C_{\mu,\nu}\, e_\mu \otimes e_\nu. \tag{2.9}$$

We know that $\frac{n(\ell(\lambda)-1)!}{\prod_{j\geq 1} m_j(\lambda)!}$ is the number of composition tableaux $\mathcal{T}$ with the same entries and length as λ, and one colored cell in the first row, which we denote as being black, as above. So, there is a natural combinatorial way of interpreting

$$\frac{n(\ell(\lambda)-1)!}{\prod_{j\geq 1} m_j(\lambda)!} \cdot C_{\mu,\nu}. \tag{2.10}$$

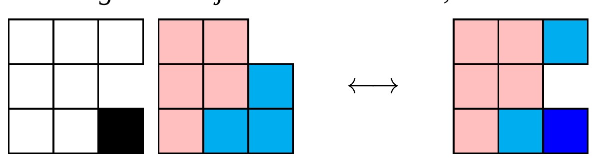
For a composition tableau $\mathcal{T}$ as above, and for a pink/cyan-colored, λ-shaped partition tableau $\mathcal{U}$ corresponding to the factor $C_{\mu,\nu}$, the product (2.10) counts all possible pairs of the form $(\mathcal{T},\mathcal{U})$. Let S_λ denote the set of all such pairs. Let us take the first row in $\mathcal{U}$, and place this row on top of the lowest row with λ_1 cells in $\mathcal{T}$, and color the cells of $\mathcal{T}$ according to the following rules: (pink + white = pink), (cyan + white = cyan), (pink + black = red), and (cyan + black = blue). We continue in this manner, by pairing the lowest white row in $\mathcal{T}$ with λ_i cells for $i > 1$ with the i^{th} row in $\mathcal{U}$, and coloring the cells of $\mathcal{T}$ accordingly. This gives us a bijection.

Example 2.3. *We can think of the pair of tableaux below as being "combined" according to the procedure indicated above, as suggested below.*

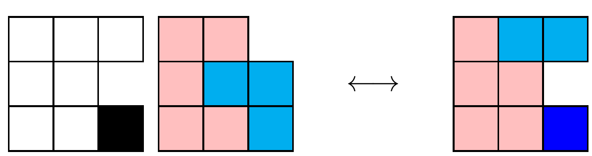


Explicitly, the bijection indicated above gives us that (2.10) counts the number of colored composition tableaux $\mathcal{V}$ satisfying the following: As a composition, $\mathcal{V}$ has the same length and entries as λ, and each row above the initial row of $\mathcal{V}$ is "split" by a possibly zero number of pink cells on the left and a possibly zero number of cyan cells on the right, whereas the first row of $\mathcal{V}$ is similarly "split" by a possibly zero number of pink/red cells on the left and a possibly zero number of cyan/blue cells on the right, modulo the condition that it must be the case that there is either exactly one blue cell but no red cells or exactly one red cell but no blue cells. Letting T_λ denote the set of all tableaux of this form, we have that S_λ and T_λ are of equal cardinality.

Example 2.4. *According to the bijection noted above, we have that*



whereas we obtain the "matching" noted below, from this one-to-one correspondence.



Let $X_n = X$ denote the set $\bigcup_{\lambda \vdash n} T_\lambda$. For a colored tableau $\mathcal{V}$ in this set, we define the *sign* associated with $\mathcal{V}$ as $(-1)^{\ell(\lambda)}$, where λ denotes the partition shape obtained by sorting the rows of $\mathcal{V}$. So, we may rewrite the equality in (2.9) as

$$\Delta(p_n) = (-1)^n \sum_{\mathcal{V} \in X} \operatorname{sgn}(\mathcal{V})\, e_{\operatorname{pr}(\mathcal{V})} \otimes e_{\operatorname{cb}(\mathcal{V})}, \tag{2.11}$$

where $\operatorname{sgn}(\mathcal{V})$ denotes the sign of $\mathcal{V} \in X$, as defined above, and $\operatorname{pr}(\mathcal{V})$ denotes the partition obtained by taking the pink/red cells in each row of $\mathcal{V}$ and sorting these resultant rows, and similarly for $\operatorname{cb}(\mathcal{V})$. So, our strategy is to determine a sign-reversing involution on X or on some subset of X that preserves or maintains the values of $\operatorname{pr}(\mathcal{V})$ and $\operatorname{cb}(\mathcal{V})$. More specifically, we want a function $\psi\colon X \to X$ such that: For $\mathcal{V} \in X$, either $\psi(\mathcal{V}) = \mathcal{V}$, or ψ maps $\mathcal{V}$ to an element in X with a different sign but such that $\operatorname{pr}(\mathcal{V}) = \operatorname{pr}(\psi(\mathcal{V}))$ and $\operatorname{cb}(\mathcal{V}) = \operatorname{cb}(\psi(\mathcal{V}))$, so that ψ encodes the cancellation of two expressions that are equal apart from having opposite signs.

Let $\mathcal{V} \in X_{n>1} = X$, and let the mapping $\psi\colon X \to X$ be defined according to the following rules.

Rule 1: First, suppose that $\mathcal{V} \in X$ has a row consisting entirely of pink cells that is immediately above a row consisting entirely of cyan/blue cells, and suppose that there is no higher row with a non-zero number of pink cells and a non-zero number of cyan cells. Let r be the highest row of pink cells satisfying the condition in the preceding sentence. In this case, define $\psi(\mathcal{V})$ as the tableau obtained from $\mathcal{V}$ by moving the cells in r into the row immediately beneath, and maintaining the requirement that pink cells must be to the left of cyan/blue cells. Conversely, if there exists a row in $\mathcal{U} \in X$ consisting of a non-zero number of pink cells and a non-zero number of blue/cyan cells, but no red cells, letting r denote the highest such row, let $\psi(\mathcal{U})$ be the tableau obtained from $\mathcal{U}$ by forming a new row immediately above r consisting of the pink cells from r, and removing the pink cells from r, and moving any cells above r in $\mathcal{U}$ upward if necessary.

Rule 2: Now, suppose that **Rule 1** does not apply for an element $\mathcal{V} \in X$. Assume that there is a red cell and at least one cyan cell in the bottom row of $\mathcal{V}$. If there exists a row of pink cells in $\mathcal{V}$, then move all of the cyan cells in the bottom row of $\mathcal{V}$ upward and immediately above the highest pink row r, and "bumping" any higher rows if necessary, and if there does not exist any row of pink cells in $\mathcal{V}$, we move all of the cyan cells in the initial row of $\mathcal{V}$ immediately upward above the first row, "bumping" any other rows of cells upwards. Conversely, and again under the assumption that **Rule 1** does not apply in this case, if there exists a red cell in the bottom row of $\mathcal{U}$, and if there are no cyan cells in the bottom row, and if there exists a row of cyan cells above the initial row, we move the lowest cyan row downward, to the bottom row.

Finally, if the above rules do not apply to an element $y \in X$, we let $\psi(y) = y$. So, the mapping $\psi\colon X \to X$ is well-defined by "default", in that we always have that $\psi(y)$ is indeed a well-defined value in the codomain of ψ for each $y \in X$.

Our strategy is to show how the following lemmas lead us toward a combinatorial way of showing how the expansion of the coproduct $\Delta(p_n)$ in (2.11) reduces to an expression equivalent to $1 \otimes p_n + p_n \otimes 1$.

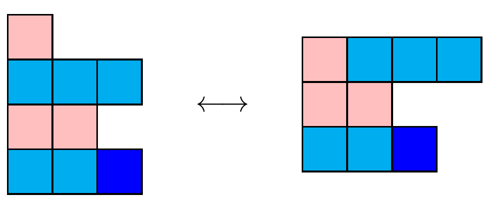
Lemma 2.1. *The mapping $\psi\colon X \to X$ is a sign-reversing involution.*

Proof. Let $\mathcal{V} \in X$. Assume that the first condition of **Rule 1**, i.e., the condition given in the first sentence of the above formulation of **Rule 1**, applies to $\mathcal{V}$. That is, $\mathcal{V}$ has a pink row immediately above a cyan/blue row such that there is no row with both pink and cyan cells above this row, again letting r denote the highest such pink row. By definition, $\psi(\mathcal{V})$ is obtained by moving the cells in row r downward to the nearest row, forming a row r' with at least one pink cell and at least one cyan/blue cell, without any such rows being above this row. Since r' is the highest row in $\psi(\mathcal{V})$ with a non-zero number of pink cells and a non-zero number of cyan/blue cells, we can see that $\psi(\psi(\mathcal{V})) = \mathcal{V}$, as ψ has the effect on $\mathcal{V}$ of moving the cells in r down to the nearest row, and ψ has the effect on $\psi(\mathcal{V})$ of moving the same cells back to their original position. A symmetric argument shows that if $\mathcal{U}$ satisfies the latter condition of **Rule 1**, we again have that $\psi(\psi(\mathcal{U})) = \mathcal{U}$. In either case, ψ has the effect of increasing/reducing the height of its argument by 1 row, thereby switching the sign associated with its argument.

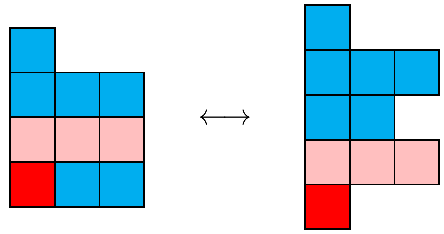
Now, suppose that **Rule 2** applies to $\mathcal{V} \in X$. So, **Rule 1** cannot apply to $\mathcal{V}$ in this case. This means that there cannot be any rows with both pink and cyan/blue cells, and there cannot be any pink rows immediately above any cyan/blue rows. So, every row above the initial row of $\mathcal{V}$ is either entirely pink or entirely cyan, and, furthermore, any such pink rows must be beneath any such cyan rows. If there is a red cell and one or more cyan cells in the initial row of $\mathcal{V}$, the mapping ψ has the effect of moving these cyan cells upward, just above any pink rows and just below any cyan rows. The mapping ψ is defined in such a way so that this function also has the effect on $\psi(\mathcal{V})$ of moving these same cyan cells back downward to their original position, i.e., so that $\psi(\psi(\mathcal{V})) = \mathcal{V}$; a symmetric argument applies if we let $\mathcal{V}$ instead satisfy the "converse" conditions for **Rule 2**. Again, the sign is reversed, since the height is increased/lowered by one row.

Finally, if **Rule 1** and **Rule 2** do not apply to $\mathcal{V} \in X$, then $\psi(\mathcal{V}) = \mathcal{V}$, so $\psi(\psi(\mathcal{V})) = \mathcal{V}$. □

Example 2.5. *The mapping defined above gives us the correspondence whereby*



according to **Rule 1**. *According to* **Rule 2**, *we have the correspondence shown below.*



Lemma 2.2. *The equality $\psi(\mathcal{V}) = \mathcal{V}$ holds if and only if $\mathcal{V}$ is entirely pink/red or entirely cyan/blue.*

Proof. ($\Longrightarrow$) If **Rule 1** or **Rule 2** applies to $\mathcal{V} \in X$, then $\psi(\mathcal{V}) \neq \mathcal{V}$, since, for example, the sign is switched. Contrapositively, if $\psi(\mathcal{V}) = \mathcal{V}$, then **Rule 1** does not apply to $\mathcal{V}$ and **Rule 2** does not apply to this tableau. Suppose that $\psi(\mathcal{V}) = \mathcal{V}$. Since **Rule 1** does not apply under this assumption, we have that no row of $\mathcal{V}$ can consist of a non-zero number of pink cells and a positive number of blue/cyan cells. So, every row of $\mathcal{V}$ above the initial row much be entirely cyan or entirely pink. As above, we also have that any pink rows above the initial row must be beneath any cyan rows above the initial row. Since **Rule 1** does not hold, the initial row of $\mathcal{V}$ cannot consist of a non-zero number of pink cells and a non-zero number of cyan/blue cells. So, either $\mathcal{V}$ has a red cell, or the initial row of $\mathcal{V}$ consists of blue/cyan cells. For the sake of clarity, we consider these two cases separately.

Case 1: Suppose that the lowest row of $\mathcal{V}$ is blue/cyan. In this case, since each row above this initial row must be completely cyan or completely pink, with any such pink rows beneath any such cyan rows, we see that there cannot be any such pink rows, because otherwise we would have a pink row immediately above a cyan/blue row, contradicting that **Rule 1** does not apply. So, if the lowest row of $\mathcal{V}$ is blue/cyan, then the entire tableau $\mathcal{V}$ must be entirely blue/cyan.

Case 2: Now, suppose that $\mathcal{V}$ has a red cell. There cannot be any cyan cells in the lowest row of $\mathcal{V}$, under our assumption that **Rule 2** does not apply. Now, recall that: Under our current assumptions, each row of $\mathcal{V}$ above the initial row must be entirely pink or entirely cyan. Since $\mathcal{V}$ has a red cell in the first row, and since there are no cyan cells this first row, there cannot be any cyan cells above the initial row, because otherwise **Rule 2** would apply. So, in this case, $\mathcal{V}$ must be entirely pink/red.

($\Longleftarrow$) We have shown that if $\psi(\mathcal{V}) = \mathcal{V}$, then $\mathcal{V}$ is either completely cyan/blue or completely pink/red. Conversely, assume that $\mathcal{V} \in X$ is either entirely cyan/blue or entirely pink/red. So, we see that **Rule 1** cannot apply, because in order for this rule to apply, $\mathcal{V}$ would have to have a pink row above a cyan/blue row, or a row with a non-zero number of pink cells and a non-zero number of blue/cyan cells. Also, we have that **Rule 2** cannot apply, because otherwise, we would have to have that $\mathcal{V}$ has a red cell and at least one cyan cell. So since **Rule 1** and **Rule 2** do not apply to $\mathcal{V}$, we must have that $\psi(\mathcal{V}) = \mathcal{V}$, completing our proof. $\square$

Theorem 2.1. *The formula for expanding power sum generators into the elementary basis with Faber partition polynomial coefficients, along with the identity $\Delta(e_n) = \sum_{i=0}^{n} e_i \otimes e_{n-i}$, together imply that each element in $\{p_m\}_{m\in\mathbb{N}}$ is primitive.*

Proof. The sign-reversing involution $\psi\colon X \to X$ preserves $\mathrm{cb}(\mathcal{V})$ and $\mathrm{pr}(\mathcal{V})$. So, with regard to the expression $\sum_{\mathcal{V}\in X} \mathrm{sgn}(\mathcal{V})\, e_{\mathrm{pr}(\mathcal{V})} \otimes e_{\mathrm{cb}(\mathcal{V})}$, in (2.11), if we take a member $\mathcal{V}$ from the index set X for the above sum, and if $\mathcal{V} \neq \psi(\mathcal{V})$, then the expressions $\mathrm{sgn}(\mathcal{V})\, e_{\mathrm{pr}(\mathcal{V})} \otimes e_{\mathrm{cb}(\mathcal{V})}$ and $\mathrm{sgn}(\psi(\mathcal{V}))\, e_{\mathrm{pr}(\psi(\mathcal{V}))} \otimes e_{\mathrm{cb}(\psi(\mathcal{V}))}$ are equal apart from opposing signs. In other words, since $\psi\colon X \to X$ is a sign-reversing involution, we can "match up" indices of our summation over all elements of X so that the term corresponding to $\mathcal{V} \in X$ cancels with that corresponding to $\psi(\mathcal{V}) \in X$, provided that $\mathcal{V}$ and $\psi(\mathcal{V})$ are distinct. So, from (2.11), we have that

$$\Delta(p_n) = (-1)^n \sum_{\substack{\mathcal{V}\in X \\ \psi(\mathcal{V})=\mathcal{V}}} \mathrm{sgn}(\mathcal{V})\, e_{\mathrm{pr}(\mathcal{V})} \otimes e_{\mathrm{cb}(\mathcal{V})},$$

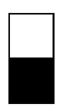
and by Lemma 2.2, we may rewrite the above equality as

$$\Delta(p_n) = \sum_{\substack{\mathcal{V}\in X \\ \mathcal{V} \text{ is cyan/blue}}} (-1)^n \mathrm{sgn}(\mathcal{V})\, e_{()} \otimes e_{\mathrm{cb}(\mathcal{V})} + \sum_{\substack{\mathcal{V}\in X \\ \mathcal{V} \text{ is pink/red}}} (-1)^n \mathrm{sgn}(\mathcal{V})\, e_{\mathrm{pr}(\mathcal{V})} \otimes e_{()},$$

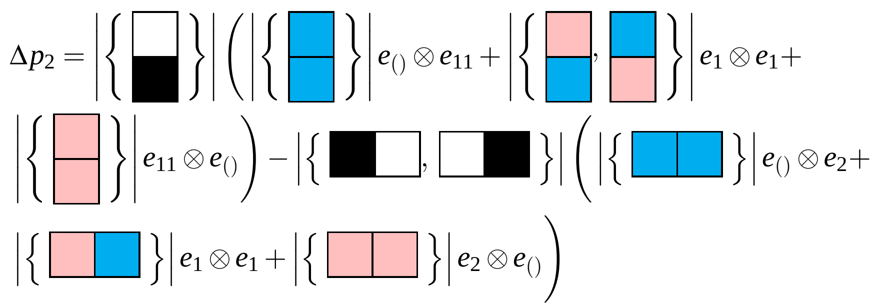
and in view of the expansion formula in (2.2) together with our combinatorial interpretation of (2.5), the right-hand side of the above equality is none other than $1 \otimes p_n + p_n \otimes 1$. □

2.1. An illustration of our sign-reversing involution on colored composition tableaux

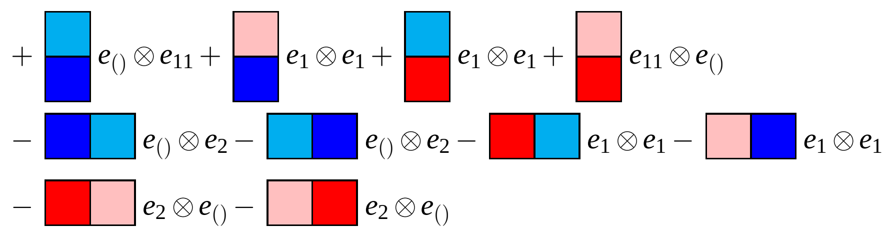
We offer a brief illustration of our involution ψ as applied to the coproduct $\Delta(p_2)$. To begin with, we write $p_2 = e_{1,1} - 2e_2$, and we note that this agrees with our combinatorial interpretation of the coefficients in (2.3), as suggested below:

$p_2 = \Big|\Big\{$  $\Big\}\Big| e_{11} - \Big|\Big\{$ ,  $\Big\}\Big| e_2.$ (2.12)

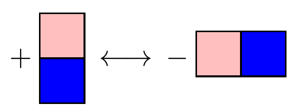
We apply Δ to both sides of the above equality. We have that $\Delta e_{11} = e_{()} \otimes e_{11} + 2e_1 \otimes e_1 + e_{11} \otimes e$ and that $\Delta e_2 = e_{()} \otimes e_2 + e_1 \otimes e_1 + e_2 \otimes e_{()}$; this agrees with our combinatorial interpretation of the coefficients in (2.8), as shown below.



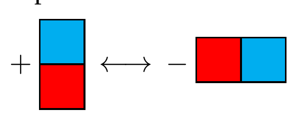
We expand the right-hand side of the above equality so as to obtain an alternating sum of simple tensors, and we abuse notation somewhat by writing the colored tableau corresponding to each such simple tensor beside each such expression, using our combinatorial interpretation of (2.10). So, we express Δp_2 as below.



At a glance, we can see that the only terms that survive are such that the associated colored tableau is entirely cyan/blue or entirely pink/red, and, modulo tensoring by 1, we obtain two equivalent copies of (2.12). According to our bijection ψ, we obtain the correspondence



by **Rule 1**, as well as the correspondence



according to **Rule 2**.

3. The remaining distinguished bases of Sym

Now, we turn our attention toward making use of what is considered to be the most important and most fundamental basis of Sym, namely the *Schur basis* $\{s_\lambda\}_{\lambda\in\mathcal{P}}$. Power sum symmetric generators may be written in terms of Schur-hooks as below:

$$p_n = \sum_{i=0}^{n} (-1)^i s_{(n-i,1^i)}. \tag{3.1}$$

This is a special case of the Murnaghan–Nakayama rule, which gives us one of the most crucial links between the subjects of representation theory and the theory of symmetric functions. We note that the important identity in (3.1) may be referred to as the *Murnaghan–Nakayama rule for a cycle*, which may be used to determine combinatorial ways of evaluating the entries in the transition matrix $M(p,s)$ more generally [4, p. 116]. This motivates our constructing new combinatorial proofs based on the Schur-hook expansion in (3.1).

We apply Δ to both sides of (3.1), so that, by linearity of this map, we have that:

$$\Delta p_n = \sum_{i=0}^{n} (-1)^i \Delta s_{(n-i,1^i)}. \tag{3.2}$$

We assume familiarity with what is meant by a *Littlewood–Richardson tableau*, or LR tableau for short. The *Littlewood–Richardson coefficient* $c^{\nu}_{\lambda,\mu}$ is defined as the number of LR tableaux of shape $\nu - \lambda$ and content μ. The famous *Littlewood-Richardson rule* $s_\lambda \cdot s_\mu = \sum_\nu c^{\nu}_{\lambda,\mu} s_\nu$ is a fundamental result in the theory of symmetric functions. This combinatorial rule is equivalent to the coproduct formula $\Delta s_\nu = \sum_{\lambda,\mu} c^{\nu}_{\lambda,\mu} s_\lambda \otimes s_\mu$ for Schur functions. We note that if ν is a hook, then λ and μ must also be hooks for $c^{\nu}_{\lambda,\mu}$ to be nonzero.

With regard to the summand in (3.2), let $Y = Y_n$ denote the set of all LR tableaux resulting from the expansion of all coproducts of the form $\Delta s_{(n-i,1^i)}$ in terms of expressions of the form $s_\lambda \otimes s_\mu$ through the LR rule. It is convenient for our purposes to let the "inner shape" of a skew tableau be denoted using unlabeled cells, as opposed to empty space. For $\mathcal{T} \in Y$, define $\phi(\mathcal{T})$ using the following procedure.

(1) If there is a 1-cell in the lowest row of $\mathcal{T}$ that is not in the first column of $\mathcal{T}$, and the tableau $\mathcal{U}$ obtained by moving this cell lexicographically into the first column of $\mathcal{T}$ (and not moving any unlabeled cells) is in Y, then $\phi(\mathcal{T}) = \mathcal{U} \in Y$;

(2) Otherwise, if there is a 1-cell in the first column of $\mathcal{T}$ that is not in the first row of $\mathcal{T}$, and the tableau $\mathcal{V}$ obtained by moving this cell lexicographically into the lowest row of $\mathcal{T}$ (and not moving any unlabeled cells) is in Y, then $\phi(\mathcal{T}) = \mathcal{V} \in Y$; and

(3) Otherwise, $\phi(\mathcal{T}) = \mathcal{T} \in Y$.

Letting $\phi\colon Y \to Y$ be as given above, it is obvious that ϕ is well-defined in the sense that $\phi(\mathcal{T}) \in Y$ for all $\mathcal{T} \in Y$. It is also not difficult to show that ϕ is an involution on Y; for the sake of brevity, this is left to the reader.

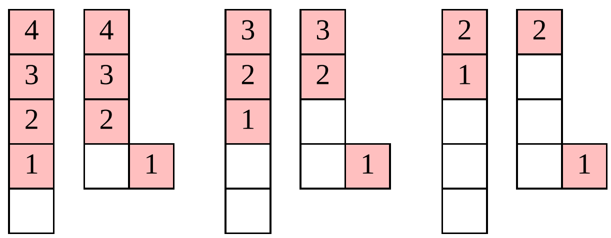
Proposition 3.1. The mapping $\phi\colon Y \to Y$ is an involution.

Again for the sake of brevity, we leave it to the reader to show that for $\mathcal{T} \in Y$, $\phi(\mathcal{T}) = \mathcal{T}$ if and only if: Either $\mathcal{T}$ has an empty interior or $\mathcal{T}$ does not have any labeled cells. This gives us a combinatorial way of showing that the expansion formula for power sum generators in terms of Schur-hooks along with the LR rule together imply that each element in $\{p_n\}_{n\in\mathbb{N}}$ is primitive: Since $\phi\colon Y \to Y$ is a sign-reversing involution with respect to the summation $\sum_{i=0}^{n}(-1)^i \Delta s_{(n-i,1^i)}$, this summation may be simplified as

$$\sum_{i=0}^{n}(-1)^i \Delta s_{(n-i,1^i)} = s_{()} \otimes \sum_{i=0}^{n}(-1)^i \Delta s_{(n-i,1^i)} + \sum_{i=0}^{n}(-1)^i \Delta s_{(n-i,1^i)} \otimes s_{()}$$

giving us that $\Delta p_n = 1 \otimes p_n + p_n \otimes 1$, as desired.

Example 3.1. *The sign-reversing involution $\phi_5 \colon Y_5 \to Y_5$ is illustrated below. Some examples of pairs of elements $\mathcal{T}_1$ and $\mathcal{T}_2$ in Y_5 such that $\phi_5(\mathcal{T}_1) = \mathcal{T}_2$ and $\phi_5(\mathcal{T}_1) \neq \mathcal{T}_2$ are illustrated beside each other below.*



Using the h-basis of Sym, as applied to the main problem of our article, we would obtain essentially the same results as in Section 2; we can formalize this idea using the morphism $\omega \colon \mathsf{Sym} \to \mathsf{Sym}$ whereby $\omega(h_\lambda) = e_\lambda$, or, equivalently, $\omega(p_k) = (-1)^{k-1} p_k$. Making use of the same notation as in (2.1), let

$$m_\mu = \sum_{\lambda \vdash |\mu|} E_{\mu,\lambda} p_\lambda / z_\lambda, \tag{3.3}$$

writing $z_\lambda := 1^{m_1(\lambda)} m_1(\lambda)! 2^{m_2(\lambda)} m_2(\lambda)! \cdots$ [11]. The family $\{m_\mu\}_{\mu \in \mathcal{P}}$ is one of the distinguished bases of Sym: The *monomial basis*. Adopting notation from [5], we have that $p_\lambda = \sum_\mu b_{\lambda,\mu} m_\mu$, where the coefficient $b_{\lambda,\mu}$ is equal to the number of ways of partitioning the entries of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ into blocks such that the sums of these blocks produce μ [5]. So, we have that $p_n = m_n$ for all indices n. However, by taking the coproduct of both sides of this equality, we immediately have that $\Delta p_n = 1 \otimes m_n + m_n \otimes 1$ by the identity $\Delta m_\lambda = \sum_{\substack{\mu,\nu \\ \mu \uplus \nu = \lambda}} m_\mu \otimes m_\nu$, so we do not obtain any cancellation. Noting the very close resemblance of the following formula to (3.3), we write

$$f_\mu = \sum_{\lambda \vdash |\mu|} (-1)^{|\lambda| - \ell(\lambda)} E_{\mu,\lambda} p_\lambda / z_\lambda, \tag{3.4}$$

giving us the *forgotten basis* $\{f_\lambda\}_{\lambda \in \mathcal{P}}$. However, we encounter the same kind of situation as before, as with the monomial basis, as one might expect, in consideration as to the close similarity between (3.3) and (3.4). To further explore combinatorial aspects about primitive elements in Sym, one might consider using the primitive family $\{m_n : n \in \mathbb{N}\}$, by expanding the members of this set into the usual bases of Sym, and applying the coproduct and simplifying, but since $m_n = p_n$ for all $n \in \mathbb{N}$, we would reproduce the above proofs.

We have successfully demonstrated how the primitivity of power sum generators in Sym may be interpreted in a combinatorial way with our bijective proof for Theorem 2.1. The results that we have introduced in this article, along with references as in [2, 3], inspire us to continue with our explorations into the development of bijective tools that elucidate the structure of bialgebras that have bases indexed by combinatorial objects.

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