AN INVERSE PROBLEM FOR STURM-LIOUVILLE TYPE DIFFERENTIAL EQUATION WITH A CONSTANT DELAY

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Abstract. The topic of this paper is a direct and inverse boundary spectral problems defined by the second order differential equation with constant delay

\[-y''(x) + q(x)y(x - \tau) = \lambda y(x)\]

where \(\tau \in \left[\frac{\pi}{2}, \pi\right)\) and \(q(x) \in L^2[0, \pi]\) and boundary conditions \(y'(0) - hy(0) = 0\) and \(y'(\pi) + H y(\pi) = 0\).

We will establish properties of the spectral characteristics and research the inverse problem of recovering operator parameters from their spectra obtained by varying right side boundary condition. We will prove that the potential, and boundary conditions are uniquely determined by the two spectra. We will also prove that the delay is recovered from one spectrum.

1. Introduction

The inverse problems in the spectral theory of operators, especially differential operators, have been studied since the 1930s. Ambarzumjan’s paper [1] is the first paper in this area, while book [2] gives deeper insight in this topic. Introducing a deviating argument into this problem, opened space for many mathematicians to further examine a whole new set of equations. A separate chapter of these studies deals with the inverse problem related to the boundary problems of the generated equations with a constant delay. In this paper we study differential operators of Sturm-Liouville-type generated with second order differential equations with a constant delay. We solve the inverse spectral problem for these operators when \(\tau \in \left[\frac{\pi}{2}, \pi\right)\).

The inverse problem of classical Sturm-Liouville operators is fully solved. The solution can be found in [4]. The methods used for solving the classical problem (transformation operator method, method of spectral mappings and others) do not

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give solution for the problems with a constant delay. Because of that, the inverse problem of second order differential operators with constant delay is still not solved.

In this paper we study the spectral boundary problem defined by the following equation

\[-y''(x) + q(x)y(x - \tau) = \lambda y(x),\]  \hspace{1cm} (1)

and the boundary conditions

\[y'(0) - hy(0) = 0.\]  \hspace{1cm} (2)

\[y'(\pi) + Hy(\pi) = 0,\]  \hspace{1cm} (3)

We will also take

\[q(x) \equiv 0, \text{ for } x \in [0, \tau).\]  \hspace{1cm} (4)

The case of Dirichlet boundary conditions and constant delay is solved in [9] and this paper is its natural continuation.

The quadruplet \((q(x), h, H, \tau)\) defines the boundary spectral problem defined by (1) – (4). It can be shown that the equation (1) with the boundary condition (2) is equivalent with the Volterra integral equation

\[y(x, z) = z \cos \pi z + h \sin \pi z + \frac{1}{z} \int_0^x q(t) \sin z(x - t) y(t - \tau, z) dt,\]  \hspace{1cm} (5)

where \(z = \sqrt{\lambda} \).

Having in mind that \(\frac{\pi}{2} \leq \tau < \pi\) using the step method we obtain the solution of equation (5)

\[y(x) = z \cos zx + h \sin zx + \int_0^x q(t) \sin z(x - t) \cos z(t - \tau) dt\]  \hspace{1cm} (6)

\[+ \frac{h}{z} \int_0^x q(t) \sin z(x - t) \sin z(t - \tau) dt.\]

From (6) we easily get

\[y'(x) = -z^2 \sin zx + hz \cos zx + z \int_0^x q(t) \cos z(x - t) \cos z(t - \tau) dt\]  \hspace{1cm} (7)

\[+ h \int_0^x q(t) \cos z(x - t) \sin z(t - \tau) dt.\]
Introducing the following functions

\[ a_{s^2}(x, z) = \int_{\tau}^{x} q(t) \sin(z(x - t)) \sin(z(t - \tau))dt, \]

\[ a_{sc}(x, z) = \int_{\tau}^{x} q(t) \sin(z(x - t)) \cos(z(t - \tau))dt, \]

\[ a_{cs}(x, z) = \int_{\tau}^{x} q(t) \cos(z(x - t)) \sin(z(t - \tau))dt, \]

\[ a_{c^2}(x, z) = \int_{\tau}^{x} q(t) \cos(z(x - t)) \cos(z(t - \tau))dt, \]

and inserting (6) and (7) into the condition (3), we obtain the characteristic function \( F(z) \) of the boundary value problem defined by (1) – (4)

\[
F(z) = \left( -z + \frac{hH}{z} \right) \sin z\pi + (h + H) \cos z\pi + a_{c^2}(z) \]

\[
+ \frac{h}{z} a_{cs}(z) + \frac{H}{z} a_{sc}(z) + \frac{hH}{z^2} a_{s^2}(z).
\]

Here we wrote \( a_{s^2}(z) \) instead of \( a_{s^2}(\pi, z) \), \( a_{sc}(z) \) instead of \( a_{sc}(\pi, z) \), etc.

Transforming integrals \( a_{s^2}(z) \), \( a_{sc}(z) \), \( a_{cs}(z) \) and \( a_{c^2}(z) \) we get

\[
a_{s^2}(z) = \frac{1}{2} (-\mathcal{I}_1 \cos(z(\pi - \tau)) + \tilde{a}_c(z)),
\]

\[
a_{cs}(z) = \frac{1}{2} (\mathcal{I}_1 \sin(z(\pi - \tau)) - \tilde{a}_s(z)),
\]

\[
a_{sc}(z) = \frac{1}{2} (\mathcal{I}_1 \sin(z(\pi - \tau)) + \tilde{a}_s(z)),
\]

\[
a_{c^2}(z) = \frac{1}{2} (\mathcal{I}_1 \cos(z(\pi - \tau)) + \tilde{a}_c(z)),
\]

where

\[
\tilde{a}_c(z) = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \tilde{q}(\theta) \cos(z(\pi - 2\theta))d\theta, \quad \tilde{a}_s(z) = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \tilde{q}(\theta) \sin(z(\pi - 2\theta))d\theta,
\]

\[
\tilde{q}(\theta) = \begin{cases} 
0, & 0 \leq \theta < \frac{\pi}{2} \\
q(\theta + \frac{\pi}{2}), & \frac{\pi}{2} \leq \theta \leq \pi - \frac{\pi}{2} \\
0, & \pi - \frac{\pi}{2} < \theta \leq \pi
\end{cases}
\]

and

\[
\mathcal{I}_1 = \tilde{a}_c(0) = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \tilde{q}(\theta)d\theta = \int_{\tau}^{x} q(t)dt.
\]
Now, characteristic function can be written as follows
\[
F(z) = \left( -z + \frac{hH}{z} \right) \sin \pi z + (h + H) \cos \pi z + \frac{1}{2} (\mathcal{I}_1 \cos z(\pi - \tau) + \tilde{a}_c(z)) \\
+ \frac{h + H}{2z} \mathcal{I}_1 \sin z(\pi - \tau) + \frac{H - h}{2z^2} a_s(z) + \frac{hH}{2z^2} (\mathcal{I}_1 \cos z(\pi - \tau) + \tilde{a}_c(z)).
\]

(10)

Now, we will find asymptotic behavior of its zeros. Firstly, we will find functions \( C_1(n) \) and \( C_2(n) \) so that asymptotic of zeros of characteristics function \( F(z) \) can be written in the form
\[
z_n = n + \frac{C_1(n)}{n} + \frac{C_2(n)}{n^2} + o \left( \frac{C_2(n)}{n^2} \right).
\]

Using elementary trigonometric transformations it can be easily shown that the following lemma holds.

**Lemma 1.** When \( n \to \infty \) following asymptotic behavior holds
\[
-\frac{z_n}{n} \sin \pi z_n = (-1)^{n+1} \left( C_1(n) \pi + \frac{C_2(n)}{n} + o \left( \frac{C_2(n)}{n} \right) \right),
\]
\[
\frac{1}{z_n} \sin \pi z_n = o \left( \frac{C_1(n) \pi}{n^2} \right),
\]
\[
\cos \pi z_n = (-1)^n + O \left( \frac{C_1(n) \pi}{n} \right),
\]
\[
\cos z_n(\pi - \tau) = (-1)^n \cos n\tau + (-1)^n C_1(n)(\pi - \tau) \sin n\tau + o \left( \frac{C_1(n)}{n} \right),
\]
\[
\tilde{a}_c(z_n) = (-1)^n \tilde{a}_2n + O \left( \frac{\tilde{a}_2n}{n} \right),
\]
\[
\sin z_n(\pi - \tau) = (-1)^n \sin n\tau + O \left( \frac{C_1(n)}{n} \right),
\]
\[
\tilde{a}_s(z_n) = (-1)^{n+1} \tilde{b}_2n + O \left( \frac{\tilde{b}_2n}{n} \right)
\]

where
\[
\tilde{a}_{2n} = \int_{\frac{\pi}{2}}^{\pi - \frac{\pi}{2}} \tilde{q}(\theta) \cos 2n\theta \, d\theta, \quad \tilde{b}_{2n} = \int_{\frac{\pi}{2}}^{\pi - \frac{\pi}{2}} \tilde{q}(\theta) \sin 2n\theta \, d\theta.
\]

By using this lemma together with \( \lambda_n = z_n^2 \), it is easily shown that the following theorem holds
Theorem 1. Eigenvalues of Sturm-Liouville spectral problem with constant delay $(q(x), h, H, \tau)$ have following asymptotic behavior
\[
\lambda_n = n^2 + C_0 + \frac{I_1}{\pi} \cos n\tau + \frac{1}{\pi} a_{2n} + \frac{1}{n} \left( C_1 \sin n\tau + C_2 \sin 2n\tau \right) + o\left( \frac{1}{n} \right), \quad n \to \infty, \quad (11)
\]
where
\[
C_0 = \frac{2}{\pi} (h + H), \quad C_1 = -\frac{(H + h)\tau}{\pi^2} I_1, \quad C_2 = \frac{\pi - \tau}{4\pi^2} I_1.
\]

From (10) it is easily deduced that the characteristic function $F(z)$ is a whole function, even function and that it has unity growth in respect to variable $z$, so it can be written in the following form (applying Hadamard factorization theorem)
\[
F(z) = \pi \left( \lambda_0 - z^2 \right) \prod_{n=1}^{\infty} \frac{\lambda_n - z^2}{n^2}, \quad (12)
\]
which will be used in the following section.

2. Main results

In this section we will show that from the two spectra \( \{\lambda_{n[j]}\}_{n \in \mathbb{N}_0} \), that match problems \((q(x), h, H_j, \tau), \ (j = 1, 2)\) respectively, the inverse problem is uniquely solved.

From the main result of previous section we know that eigenvalues $\lambda_{n[j]}$ have asymptotic behavior
\[
\lambda_{n[j]} = n^2 + C_0^{(j)} + \frac{I_1}{\pi} \cos n\tau + \frac{1}{\pi} a_{2n} + \frac{1}{n} \left( C_1^{(j)} \sin n\tau + C_2 \sin 2n\tau \right) + o\left( \frac{1}{n} \right), \quad n \to \infty. \quad (13)
\]

Analogously as in [9] it can be shown that the following theorem is valid

Theorem 2. If $\{\lambda_n\}_{n \in \mathbb{N}_0}$ is spectrum of boundary value problem $(q(x), h, H, \tau)$, then
\[
\tau = \lim_{n \to \infty} \arccos \frac{\mu_n}{2}, \quad (14)
\]
where
\[
\mu_n = \frac{(\lambda_{n+2} - (n+2)^2) - (\lambda_{n-2} - (n-2)^2)}{(\lambda_{n+1} - (n+1)^2) - (\lambda_{n-1} - (n-1)^2)}.
\]

From (13) we have
\[
C_0^{(2)} - C_0^{(1)} = \lim_{n \to \infty} (\lambda_n^{(2)} - \lambda_n^{(1)}).
\]
Because
\[
C_0^{(j)} = \frac{2}{\pi} (h + H_j),
\]
we get
\[ H_2 - H_1 = \frac{\pi}{2} (C_0^{(2)} - C_0^{(1)}) = \frac{\pi}{2} \lim_{n \to \infty} (\lambda_n^{(2)} - \lambda_n^{(1)}). \] (15)

It is possible to find subsequences \( n_k^{(1)} \) i \( n_k^{(2)} \) of sequence \( n \) for which
\[ \cos n_k^{(1)} \tau \neq 0, \quad \cos n_k^{(2)} \tau \neq 0 \]
and
\[ |\cos n_k^{(1)} \tau - \cos n_k^{(2)} \tau| \geq \delta > 0, \quad \forall k. \]

From (13) we get
\[ I_1 = \pi \lim_{k \to \infty} \left( \frac{\lambda_{n_k}}{\cos n_k^{(1)} \tau - \cos n_k^{(2)} \tau} \right). \] (16)

From (13), bearing in mind that we already determined \( \tau, H_2 - H_1 \) and \( I_1 \) we get
\[ C_0^{(j)} = \lim_{k \to \infty} \left( \frac{\lambda_{n_k} - (n_k)^2}{\cos n_k^{(1)} \tau - \cos n_k^{(2)} \tau} \right), \quad (j = 1, 2). \] (17)

Let’s notice that \( C_0^{(j)} \) can be recovered without previous knowledge of \( I_1 \). Namely, we have
\[ C_0^{(j)} = \lim_{k \to \infty} \left( \frac{\lambda_{n_k} - (n_k)^2}{\cos n_k^{(1)} \tau - \cos n_k^{(2)} \tau} \right), \quad (j = 1, 2). \]

Before we recover \( h, H_1 \) and \( H_2 \) we must write characteristic function \( F(z) \) in a somewhat different form.

From (8) we have
\[ F_2(z) - F_1(z) = \frac{h}{z} (H_2 - H_1) \sin \pi z + (H_2 - H_1) \cos \pi z \]
\[ + \frac{H_2 - H_1}{2z} I_1 \sin z(\pi - \tau) + o\left(\frac{1}{z^2}\right). \] (18)

Solving for \( h \) we get
\[ h = \frac{1}{\sin \pi z} \left[ \frac{F_2(z) - F_1(z)}{H_2 - H_1} z - z \cos \pi z - \frac{I_1 \sin z(\pi - \tau)}{2} \right] + o\left(\frac{1}{z}\right). \] (19)

Replacing \( z = 2k + \frac{1}{2} \) in (19) and letting \( k \to \infty \) we obtain
\[ h = \lim_{k \to \infty} \frac{1}{H_2 - H_1} \left( F_2(2k + \frac{1}{2}) - F_1(2k + \frac{1}{2}) \right) (2k + \frac{1}{2}) - \frac{I_1}{2} \cos \left( 2k + \frac{1}{2} \right) \tau. \] (20)

After we have recovered \( h \) from (20) we recover \( H_1 \) and \( H_2 \) by
\[ H_j = \frac{\pi}{2} C_0^{(j)} - h \quad (j = 1, 2). \] (21)
Let us define the following functions
\[
A(z) = \frac{H_2 F_1(z) - H_1 F_2(z)}{H_2 - H_1} + z \sin \pi z - h \cos \pi z,
\]
\[
B(z) = z \frac{F_2(z) - F_1(z)}{H_2 - H_1} - h \sin \pi z - z \cos \pi z.
\]
From (8) we have
\[
A(z) = \frac{1}{2} [I_1 \cos z(\pi - \tau) - \tilde{a}_s(z)] + \frac{h}{2z} [I_1 \sin z(\pi - \tau) - \tilde{a}_c(z)]
\]
\[
B(z) = \frac{1}{2} [I_1 \sin z(\pi - \tau) - \tilde{a}_s(z)] - \frac{h}{2z} [I_1 \cos z(\pi - \tau) - \tilde{a}_c(z)]
\]
(22)

Integrating by parts we eliminate \( z \) from denominators and get
\[
\frac{\tilde{a}_s(z)}{2z} = \frac{\sin z(\pi - \tau)}{-2z} I_1 + J \tilde{a}_c(z), \quad \frac{\tilde{a}_c(z)}{2z} = -\frac{\cos z(\pi - \tau)}{2z} I_1 + J \tilde{a}_s(z).
\]

Now (22) turns into
\[
A(z) = \frac{1}{2} (I_1 \cos z(\pi - \tau) + \tilde{a}_c(z)) + h \left[ \frac{I_1}{2z} \sin z(\pi - \tau) + \frac{I_1}{2z} \sin z(\pi - \tau) - J \tilde{a}_c(z) \right]
\]
\[
B(z) = \frac{1}{2} (I_1 \sin z(\pi - \tau) + \tilde{a}_s(z)) + h \left[ -\frac{I_1}{2z} \cos z(\pi - \tau) + \frac{I_1}{2z} \cos z(\pi - \tau) - J \tilde{a}_s(z) \right].
\]

Let
\[
A_1(z) = 2A(z) - h I_1 \frac{\sin z(\pi - \tau)}{z} - I_1 \cos z(\pi - \tau), \quad B_1(z) = 2B(z) - I_1 \sin z(\pi - \tau)
\]
Notice that \( B_1(z) \to 0 \), when \( z \to 0 \).

So we get
\[
A_1(z) = \tilde{a}_c(z) - 2h J \tilde{a}_c(z), \quad B_1(z) = \tilde{a}_s(z) - 2h J \tilde{a}_s(z)
\]

Taking \( z = m, m \in \mathbb{N}_0 \) we obtain system
\[
A_1(m) = \tilde{a}_c(m) - 2h J \tilde{a}_c(m), \quad B_1(m) = \tilde{a}_s(m) - 2h J \tilde{a}_s(m)
\]
or
\[
A_1(m) = (-1)^m \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} q(\theta) \cos 2m\theta d\theta - 2h(-1)^m \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \int_{\frac{\pi}{2}}^{\theta} q(\theta_1) d\theta_1 \right) \cos 2m\theta d\theta
\]
\[
B_1(m) = (-1)^{m+1} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} q(\theta) \sin 2m\theta d\theta - 2h(-1)^{m+1} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \int_{\frac{\pi}{2}}^{\theta} q(\theta_1) d\theta_1 \right) \sin m(\pi - 2\theta) d\theta.
\]
Let us define sequences $A_{2m}$ and $B_{2m}$ in the following way
\begin{equation}
A_{2m} = (-1)^m \frac{2}{\pi} A_1(m) = \tilde{a}_{2m} - 2h\mathcal{J}\tilde{a}_{2m},
B_{2m} = (-1)^{m+1} \frac{2}{\pi} B_1(m) = \tilde{b}_{2m} - 2h\mathcal{J}\tilde{b}_{2m}.
\end{equation}

System (23) relates cosine Fourier coefficients of the functions $\tilde{q}(\theta)$, $\theta \int_{\frac{\pi}{2}}^{\theta} \tilde{q}(\theta_1) d\theta_1$ with $A_{2m}$ and sine Fourier coefficients of the functions $\tilde{q}(\theta)$, $\frac{\partial}{\partial \theta} \int_{\frac{\pi}{2}}^{\theta} \tilde{q}(\theta_1) d\theta_1$ with $B_{2m}$.

**Lemma 2.** Sequences $\{A_{2m}\}_{m \in \mathbb{N}_0}$ and $\{B_{2m}\}_{m \in \mathbb{N}_0}$ are Fourier coefficients of some function $f \in L^2[0, \pi]$.

From this lemma we have
\[
f(x) = \sum_{m=1}^{\infty} A_{2m} \cos 2mx + B_{2m} \sin 2mx.
\]

Finally, we conclude that the system (23) is equivalent to integral equation
\begin{equation}
\tilde{q}(x) = f(x) + 2h \int_{0}^{x} \tilde{q}(\theta) d\theta.
\end{equation}

Equation (24) is linear Volterra integral equation of second type. Notice that the kernel of this equation is constant $2h$. From [8] it follows that its unique solution is given in the form
\[
\tilde{q}(x) = f(x) + 2h \int_{0}^{x} e^{2h(x-t)} f(t) dt.
\]

Herewith, we have solved the inverse problem.

**References**


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