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TENSOR ALGEBRAS OF BIMODULES AND THEIR REPRESENTATIONS

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Dedicated to the memory of Professor Marc Krasner

ABSTRACT. We study $(D, \mathcal{O})$-species, which are a special case of species introduced by Yu. A. Drozd (1980). The representations of $(D, \mathcal{O})$-species and modules over the corresponding tensor algebras of bimodules are considered. We find necessary and sufficient conditions on a special kind of $(D, \mathcal{O})$-species under which they are of bounded representation type. The conditions are given in terms of Dynkin diagrams and diagrams with weights. The connection of these $(D, \mathcal{O})$-species and corresponding tensor algebras with right hereditary semiperfect and semidistributive rings are studied.

1. INTRODUCTION

Quivers and species play a key role in the representation theory of associative algebras and rings. The notion of species was introduced by P. Gabriel in [7] where he characterized a special case of K-species of finite representation type. His result was extended by Dlab and Ringel [1], [2] to arbitrary K-species and valued quivers.

Drozd [6] generalized the notion of species as follows. Let I be a finite index set. A species is a finite collection $\mathfrak{L} = (A_i, {}_iM_j)_{i,j \in I}$, where all A_i are prime rings and all ${}_iM_j$ are A_i-A_j-bimodules. If all $A_i = F_i$ are division rings we obtain a species in the sense of P. Gabriel. If each division ring F_i is finite dimensional and central over a common commutative subfield K that

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acts centrally on ${}_iM_j$, we obtain a K-species. If each $A_i = D$ is a division ring we say that $\mathfrak{L}$ is a D-species.

We associate a tensor algebra $\mathfrak{T}(\mathfrak{L})$ to a species $\mathfrak{L} = (A_i, {}_iM_j)_{i,j\in I}$ in the following way. Let $B = \prod_{i\in I} A_i$, and $M = \oplus_{i,j\in I}\, {}_iM_j$. Then B is a ring and M naturally becomes a (B,B)-bimodule. Therefore we can consider the graded ring

$$\mathfrak{T}_B(M) = \bigoplus_{i=0}^{\infty} T_i, \tag{1.1}$$

where $T_0 = B$ and $T_{i+1} = T_i \otimes_B M$ $(i > 0)$ with component-wise addition and the multiplication induced by taking tensor products. Then $\mathfrak{T}_B(M)$ is a tensor algebra of the B-bimodule M and it is called the **tensor algebra** of the species $\mathfrak{L}$. Denote it by $\mathfrak{T}(\mathfrak{L})$.

The **quiver** $\Gamma(\mathfrak{L})$ of a species $\mathfrak{L}$ is defined as the directed graph whose vertices are indexed by the numbers $i = 1, \ldots, n$, and there is an arrow from the vertex i to the vertex j if and only if ${}_iM_j \neq 0$.

Dlab and Ringel [2, Proposition 10.1] proved that the category $\operatorname{Rep}\mathfrak{L}$ of all representations of a K-species $\mathfrak{L}$ and the category $\operatorname{Mod}_r \mathfrak{T}(\mathfrak{L})$ of all right $\mathfrak{T}(\mathfrak{L})$-modules are equivalent. They also characterized K-species of finite type in terms of Dynkin diagrams [2, Theorem B].

A finite dimensional algebra is said to be of **finite type** if there is only a finite number of indecomposable finite dimensional A-modules. Dlab and Ringel [2, Theorem C] proved that a finite dimensional K-algebra A is a hereditary algebra of finite type if and only if A is Morita equivalent to the tensor algebra $\mathfrak{T}(\mathfrak{L})$, where $\mathfrak{L}$ is a K-species of finite type.

Dowbor, Ringel and Simson [4, Theorem 2] generalized this result to hereditary Artinian rings by using valued graphs with valuations. They proved that a hereditary Artinian basic ring is of finite type if and only if the corresponding valued graph is a disjoint union of Coxeter diagrams.

By $\mathcal{O}$-species we mean those species $(A_i, {}_iM_j)_{i,j\in I}$ (in the sense of Yu. A. Drozd) in which all A_i are prime hereditary Noetherian semiperfect rings. We study the representations of $\mathcal{O}$-species and modules over the corresponding tensor algebras.

By Warfield [17], a ring A is of **bounded representation type** (b.r.t., for short) if the number of generators required for indecomposable finitely presented right A-modules is bounded. In analogous way one can introduce $\mathcal{O}$-species of bounded representation type. In this paper we consider $(D,\mathcal{O})$-species that are the special case of $\mathcal{O}$-species. We find necessary and sufficient conditions on a $(D,\mathcal{O})$-species of a special type under which it is of bounded representation type. We describe such $(D,\mathcal{O})$-species in terms of

Dynkin diagrams and diagrams with weights. The problem of the description of their representations is reduced to flat mixed matrix problems over discrete valuation rings and their common skew field of fractions D.

Matrix problems, i.e. the problems of reducing a family of matrices by some family of admissible transformations, arise in many problems of representation theory. The general definition of a matrix problem was given by Roiter [16] over a field and then was generalized by Drozd [5] to matrix problems over rings.

In this paper, we also study the connection of representations of $(D,\mathcal{O})$-species with modules over right hereditary semiperfect and semidistributive rings and describe these rings of bounded representation type.

We use the notions and results from [11], [12] and [13]. Throughout this paper all rings are assumed to be associative with $1 \neq 0$ and all modules are assumed to be unital.

2. $\mathcal{O}$-SPECIES AND TENSOR ALGEBRAS

In this section we consider $\mathcal{O}$-species that are a special case of species as introduced by Drozd [6].

Let $\{\mathcal{O}_i\}$ be a family of discrete valuation rings (not necessarily commutative) $\mathcal{O}_i$ with Jacobson radicals R_i and skew fields of fractions D_i for $i = 1, 2, \ldots, k$, and let $\{D_j\}$ for $j = k+1, \ldots, n$ be a family of skew fields. Let $\{n_1, n_2, \ldots, n_k\}$ be a set of natural numbers. Write

$$H_{n_i}(\mathcal{O}_i) = \begin{pmatrix} \mathcal{O}_i & \mathcal{O}_i & \cdots & \mathcal{O}_i \\ R_i & \mathcal{O}_i & \cdots & \mathcal{O}_i \\ \vdots & \vdots & \ddots & \vdots \\ R_i & R_i & \cdots & \mathcal{O}_i \end{pmatrix}$$

that is a subring of the matrix ring $M_{n_i}(D_i)$. It is easy to see that each $H_{n_i}(\mathcal{O}_i)$ is a Noetherian serial prime hereditary ring. Write $A_i = H_{n_i}(\mathcal{O}_i)$ for $i = 1, 2, \ldots, k$, and $A_j = D_j$ for $j = k+1, \ldots, n$. By the Goldie theorem, there exists a classical ring of fractions $\tilde{A}_i = M_{n_i}(D_i)$ for $i = 1, 2, \ldots, k$ and $\tilde{A}_j = D_j$ for $j = k+1, \ldots, n$.

Definition 2.1. Let $I = \{1, 2, \ldots, n\}$. An $\mathcal{O}$-**species** is a family $\Omega = (A_i, {}_iM_j)_{i,j\in I}$, where $A_i = H_{n_i}(\mathcal{O}_i)$, for $i = 1, \ldots, k$, $A_j = D_j$ for $j = k+1, \ldots, n$, and each ${}_iM_j$ is an $(\tilde{A}_i, \tilde{A}_j)$-bimodule that is finite dimensional both as a left D_i-vector space and as a right D_j-vector space.

An $\mathcal{O}$-species Ω is said to be a $(D,\mathcal{O})$-**species** if all $\mathcal{O}_i$ have a common skew field of fractions D, i.e. all D_i are equal to a fixed skew field D $(i = 1, \ldots, n)$.

In Section 1 we associate with an $\mathcal{O}$-species Ω the quiver $\Gamma(\Omega)$. A vertex i is said to be **marked** if $A_i = H_{n_i}(\mathcal{O}_i)$. Let $J = \{1, \ldots, k\} \subset I$ be the set of all marked points. An $\mathcal{O}$-species Ω is called **min-marked** if all marked vertices of it are minimal and are not connected between themselves in $\Gamma(\Omega)$, i.e. ${}_iM_j = 0$ for $i \in I \setminus J$, $j \in J$, and ${}_iM_j = 0$ for all $i, j \in J$.

An $\mathcal{O}$-species is **simply connected** if the underlying graph of $\Gamma(\Omega)$ is a tree. A $(D, \mathcal{O})$-species Ω is said to be **weak** if Ω is min-marked and all A_i are $\mathcal{O}_i$ or D.

Definition 2.2. A **representation** $V = (U_i, V_r, {}_j\varphi_i, {}_j\psi_r)$ of an $\mathcal{O}$-species $\Omega = (A_i, {}_iM_j)_{i,j\in I}$ is a family of right A_i-modules U_i $(i = 1, 2, \ldots, k)$, a family of right D_r-vector spaces V_r $(r = k+1, \ldots, n)$, and of D_j-linear maps:

$$ {}_j\varphi_i : U_i \otimes_{A_i} {}_iM_j \longrightarrow V_j \tag{2.1}$$

for each $i = 1, 2, \ldots, k$ and $j = k+1, \ldots, n$, and

$$ {}_j\psi_r : V_r \otimes_{D_r} {}_rM_j \longrightarrow V_j \tag{2.2}$$

for each $r, j = k+1, \ldots, n$.

Two representations $V = (U_i, V_r, {}_j\varphi_i, {}_j\psi_r)$ and $V' = (U'_i, V'_r, {}_j\varphi'_i, {}_j\psi'_r)$ are called **equivalent** if there is a family of isomorphisms $\{\alpha_i\}$ of A_i-modules from U_i to U'_i and a family of isomorphisms $\{\beta_r\}$ of D_r-vector spaces from V_r to V'_r such that for each $i = 1, \ldots, k$ and $r, j = k+1, \ldots, n$ the following equalities hold :

$$ {}_j\varphi'_i(\alpha_i \otimes 1) = \beta_j \cdot {}_j\varphi_i, \tag{2.3}$$

$$ {}_j\psi'_r(\beta_r \otimes 1) = \beta_j \cdot {}_j\psi_r. \tag{2.4}$$

In a natural way one can define the notions of the direct sum of representations and an indecomposable representation.

The set of all representations of an $\mathcal{O}$-species $\Omega = (A_i, {}_iM_j)_{i,j\in I}$ can be turned into a category $\mathrm{Rep}(\Omega)$, whose objects are representations $V = (U_i, V_r, {}_j\varphi_i, {}_j\psi_r)$, and a morphism from an object $V = (U_i, V_r, {}_i\varphi_j, {}_i\psi_j)$ to an object $V' = (U'_i, V'_r, {}_j\varphi'_i, {}_j\psi'_r)$ is a set of homomorphisms α_i of A_i-modules M_i to M'_i, and a set of homomorphisms β_r of D_r-vector spaces V_r to V'_r such that for each $i = 1, \ldots, k$ and $r, j = k+1, \ldots, n$ the equalities (2.5) and (2.6) hold.

For any $\mathcal{O}$-species Ω one can construct the tensor algebra of bimodules $\mathfrak{T}(\Omega)$ (see Section 1). The following statement is proved similarly to [2, Proposition 10.1].

Proposition 2.3. *Let Ω be a simply connected min-marked $\mathcal{O}$-species. Then the category $\mathrm{Rep}(\Omega)$ of all representations of Ω and the category $\mathrm{Mod}_r\mathfrak{T}(\Omega)$ of all right $\mathfrak{T}(\Omega)$-modules are naturally equivalent.*

Definition 2.4. A representation $V = (U_i, V_r, {}_j\varphi_i, {}_j\psi_r)$ of a simply connected min-marked $\mathcal{O}$-species $\Omega = (A_i, {}_iM_j)_{i,j\in I}$ is said to be finite dimensional if each U_i is a finitely generated A_i-module for $i = 1, \dots, k$ and each V_r is a finite dimensional D_r-vector space for $r = k+1, \dots, n$.

The **dimension** of a representation $V = (U_i, V_r, {}_j\varphi_i, {}_j\psi_r)$ is the number

$$d = \dim V = \sum_{i=1}^{k} d(U_i) + \sum_{r=k+1}^{n} d(V_r). \tag{2.5}$$

where $d(U_i)$ is the minimum number of generators of an A_i-module M_i, and $d(V_r) = \dim_{D_r}(V_r)$ is the dimension of a vector space V_r over D_r.

Definition 2.5. An $\mathcal{O}$-species Ω is said to be of bounded representation type if the dimensions of its indecomposable finite dimensional representations are bounded above.

These definitions and Proposition 2.3 ensure the following corollaries.

Corollary 2.6. *A simply connected min-marked $(D, \mathcal{O})$-species Ω is of bounded representation type if and only if the tensor algebra $\mathfrak{T}(\Omega)$ is of bounded representation type.*

Corollary 2.7. *Let Ω_1 be a D-species that is a subspecies of a simply connected min-marked $(D, \mathcal{O})$-species Ω. If Ω is of bounded representation type, then Ω_1 is of finite type.*

Along with a weak $(D, \mathcal{O})$-species $\Omega = (F_i, {}_iM_j)_{i,j\in I}$ one can consider a D-species $\tilde{\Omega} = (\tilde{F}_i, {}_iM_j)_{i,j\in I}$ with $\tilde{F}_i = D$, since each ${}_iM_j$ is an $(\tilde{F}_i, \tilde{F}_j)$-bimodule. Let $\mathfrak{T}(\tilde{\Omega})$ be a tensor algebra of D-species $\tilde{\Omega}$. Since $\mathfrak{T}(\tilde{\Omega})$ is an Artinian ring, it is of bounded representation type if and only if it is of finite representation type.

Proposition 2.8. *If Ω is a weak simply connected $(D, \mathcal{O})$-species of bounded representation type, then $\tilde{\Omega}$ is a D-species of finite representation type.*

Proof. Let Ω be a weak simply connected $(D, \mathcal{O})$-species with the set of marked vertices $J = \{1, 2, \dots, k\}$. Then the tensor algebra $A = \mathfrak{T}(\Omega)$ is a basic primely triangular ring with two-sided Peirce decomposition of the form

$$A = \left(\begin{array}{ccc|c} \mathcal{O}_1 & \dots & 0 & U_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \mathcal{O}_k & U_k \\ \hline 0 & \dots & 0 & T \end{array}\right) \tag{2.6}$$

where U_i is an (D,T)-bimodule $(i = 1, 2, \dots, k)$. Moreover the ring T is a tensor algebra of a species $\Omega_1 = (F_i, {}_iM_j)_{i,j \in I \setminus J}$ where $F_i = D$ for all $i \in I \setminus J$.

Since Ω is a $(D,\mathcal{O})$-species of bounded representation type, the tensor algebra $\mathfrak{T}(\Omega)$ is also of bounded representation type. Then by Proposition 2.7, T is also of bounded representation type. Since Ω_1 is a D-species, T is an Artinian ring, and so it is of finite representation type. Since Ω is simply connected, Ω_1 is also simply connected. By [15, Lemma 5.3] T is an Artinian hereditary ring.

The rest of the proof is similar to the proof of [8, Theorem III]. □

3. Representations of $(D,\mathcal{O})$-species and mixed matrix problems

The aim of this section is to show that the description of representations of weak simply connected $(D,\mathcal{O})$-species can be reduced to some flat mixed matrix problems over discrete valuation rings $\mathcal{O}_i$ for $i = 1, \dots, k$ and their common skew field of fractions D. These matrix problems generalize a flat matrix problem considered by Zavadskij and Revitskaya [18]. Some examples of such flat matrix problems were considered in [9].

Let $\mathcal{O}$ be a discrete valuation ring (DVR) with a classical division ring of fractions D.

By left $\mathcal{O}$-elementary transformations of rows of a matrix $\mathbf{T}$ with entries in D we mean transformations of two types:

(a) multiplying a row on the left by an invertible element of $\mathcal{O}$;
(b) adding a row multiplied on the left by an element of $\mathcal{O}$ to another row.

In a similar way we can define left D-elementary transformations of rows and, by symmetry, right $\mathcal{O}$-elementary and right D-elementary transformations of columns.

Elementary transformations of these type can be given by invertible elementary matrices. The automorphism of a finitely generated module P corresponding to elementary transformation is an elementary automorphism. Multiplications on the left (right) of a matrix $\mathbf{T}$ by elementary matrices correspond to elementary row (column) transformations.

By [11, Proposition 13.1.3], any invertible matrix $\mathbf{B}$ over a local ring $\mathcal{O}$ can be reduced by $\mathcal{O}$-elementary row (column) transformations on $\mathbf{B}$ to the identity matrix. By [11, Corollary 13.1.4], the matrix $\mathbf{B}$ can be decomposed into a product of elementary matrices. Moreover, by [11, Theorem 13.1.6] any automorphism of a finitely generated projective module P over a semiperfect ring A can be decomposed into a product of elementary automorphisms.

Let $\Delta = \{\mathcal{O}_i\}_{i=1,\dots,k}$ be a family of discrete valuation rings $\mathcal{O}_i$ with a common skew field of fractions D. We define the general flat matrix problem over Δ and D in the following way.

Let $\mathbf{T}$ be a block rectangular matrix with entries in D partitioned into n horizontal strips $\{\mathbf{T}_i\}_{i=1,\dots,n}$ and m vertical strips $\{\mathbf{T}^j\}_{j=1,\dots,m}$ so that each block $\mathbf{T}_i^j$ is the intersection of j-th vertical strip and i-th horizontal strip; some of these blocks may be empty.

Assume that the ring $F_{i_s} \in \Delta \cup D$ corresponds to the i-th horizontal strip $\mathbf{T}_i$ and the ring $F_{j_t} \in \Delta \cup D$ corresponds to the j-th vertical strip $\mathbf{T}^j$.

One has the following admissible transformations with the matrix $\mathbf{T}$:

1. Left F_{i_s}-elementary transformations of rows within the strip $\mathbf{T}_i$.
2. Right F_{j_t}-elementary transformations of rows within the strip $\mathbf{T}^j$.
3. Additions of rows in the strip $\mathbf{T}_j$ multiplied on the left by elements of $F_r \in \Delta \cup D$ to rows in the strip $\mathbf{T}_i$.
4. Additions of columns in the strip $\mathbf{T}^i$ multiplied on the right by elements of $F_p \in \Delta \cup D$ to columns in the strip $\mathbf{T}^j$.

Indecomposable matrices and equivalent matrices are defined in a natural way. A flat matrix problem is said to be of **finite type** if the number of non-equivalent indecomposable matrices is finite.

Definition 3.1. The vector

$$d = d(\mathbf{T}) = (d_1, d_2, \dots, d_n; d^1, d^2, \dots d^m), \tag{3.1}$$

where d_i is the number of rows of the i-th horizontal strip of $\mathbf{T}$ for $i = 1, \dots, n$ and d^j is the number of columns of the j-th vertical strip of $\mathbf{T}$ for $j = 1, \dots, m$, is called the **dimension vector** of the partition matrix $\mathbf{T}$. We set

$$\dim(\mathbf{T}) = \sum_{i=1}^{n} d_i + \sum_{j=1}^{m} d^j. \tag{3.2}$$

Definition 3.2. We say that a flat matrix problem is of **bounded representation type** if there is a constant C such that $\dim(\mathbf{X}) < C$ for all indecomposable matrices $\mathbf{X}$. Otherwise it is of **unbounded representation type**.

Let $\Omega = (A_i, {}_iM_j)_{i,j \in I}$ be a weak $(D, \mathcal{O})$-species of bounded representation type. Suppose that $V = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r)$ is an indecomposable finite dimensional representation of Ω, then M_i is a finitely generated A_i-module for $i = 1, \dots, k$ and V_r is a finite dimensional D-vector space for $r = k+1, \dots, n$. Since $A_i = \mathcal{O}_i$ is a discrete valuation ring by [13, proposition 5.4.18], any $\mathcal{O}_i$-module M_i is torsion-free and faithful. Therefore, any indecomposable representation of Ω has the following form:

$$V = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r), \tag{3.3}$$

where M_i is a free A_i-module.

Let $V = (M_i, V_r, {}_j\varphi_i, {}_j\psi_r)$ and $V' = (U_i', V_r', {}_j\varphi_i', {}_j\psi_r')$ be indecomposable finite dimensional representations of a $(D, \mathcal{O})$-species Ω. Then each M_i is a finitely generated free $\mathcal{O}_i$-module with basis $\omega_1^{(i)}, \dots, \omega_{m_i}^{(i)}$ $(i = 1, \dots, k)$; and V_r is a finite dimensional D-space with basis $\tau_1^{(r)}, \dots, \tau_{k_r}^{(r)}$ $(r = k+1, \dots, n)$. Suppose that

$$ {}_j\varphi_i(\omega_s^{(i)} \otimes 1) = \sum_{u=1}^{k_j} \tau_u^{(j)} b_{us}^{(ij)}, \tag{3.4} $$

$$ {}_j\psi_r(\tau_v^{(r)} \otimes 1) = \sum_{u=1}^{k_i} \tau_u^{(j)} a_{uv}^{(ij)}, \tag{3.5} $$

where $a_{uv}^{(ij)}, b_{us}^{(ij)} \in D$. Then the matrices $\mathbf{A}_{ij} = (a_{uv}^{(ij)})$, and $\mathbf{B}_{ij} = (b_{us}^{(ij)})$ define the representation V uniquely up to equivalence.

Let $\mathbf{U}_i \in M_{m_i}(A_i)$ be the matrix corresponding to the homomorphism α_i, and let $\mathbf{W}_i \in M_{k_i}(D)$ be the matrix corresponding to the homomorphism β_i, $i \in I$. If $\mathbf{A}'_{ij}$, $\mathbf{B}'_{ij}$ are the matrices corresponding to the equivalent representation V' then the matrices $\mathbf{U}_i$ and $\mathbf{W}_i$ are invertible, and therefore the equalities (2.5) and (2.6) have the following matrix form:

$$ \mathbf{W}_i \mathbf{B}_{ij} \mathbf{U}_j^{-1} = \mathbf{B}'_{ij}, \quad (i = 1, \dots, k; \quad j = k+1, \dots, n) \tag{3.6} $$

$$ \mathbf{W}_j \mathbf{A}_{jr} \mathbf{W}_r^{-1} = \mathbf{A}'_{jr}, \quad (j, r = k+1, \dots, n). \tag{3.7} $$

Thus, we obtain the following matrix problem of finding indecomposable finite dimensional representations of a $(D, \mathcal{O})$-species Ω.

3.1. **Main mixed matrix problem.** Let $\Delta = \{\mathcal{O}_i\}_{i=1,\dots,k}$ be a family of discrete valuation rings $\mathcal{O}_i$ with a common skew field of fractions D.

Let $\mathbf{T}$ be a block rectangular matrix with entries in D partitioned into n horizontal strips $\{\mathbf{T}_i\}_{i=1,\dots,n}$ and m vertical strips $\{\mathbf{T}^j\}_{j=1,\dots,m}$ so that each block $\mathbf{T}_i^j$ is the intersection of the j-th vertical strip and the i-th horizontal strip, some of these matrices may be empty.

One has the following admissible transformations with the matrix $\mathbf{T}$:

1. Left F_{i_s}-elementary transformations of rows within the strip $\mathbf{T}_i$, where $F_{i_s} \in \Delta \cup D$.
2. Right F_{j_t}-elementary transformations of rows within the strip $\mathbf{T}^j$, where $F_{j_t} \in \Delta \cup D$.

The admissible transformations with the matrix $\mathbf{T}$ can be given in the form $\mathbf{T} \mapsto \mathbf{XTY}$, where $\mathbf{X} = \operatorname{diag}(\mathbf{X}_1, \dots, \mathbf{X}_n)$ and $\mathbf{Y} = \operatorname{diag}(\mathbf{Y}_1, \dots, \mathbf{Y}_m)$, and all $\mathbf{X}_i$ and $\mathbf{Y}_j$ are square invertible matrices. Moreover, $\mathbf{X}_i \subset M_{m_i}(F_{i_s})$ and $\mathbf{Y}_j \subset M_{k_j}(F_{j_t})$, where $F_{i_s}, F_{j_t} \in \Delta \cup D$.

Clearly, the matrix $\mathbf{T}$ is indecomposable if and only if the corresponding representation of Ω is indecomposable. It is easy to prove the following statement.

Lemma 3.3. *A simply connected weak $(D,\mathcal{O})$-species Ω is of bounded representation type if and only if the corresponding main matrix problem is of bounded representation type.*

4. $(D,\mathcal{O})$-species of bounded representation type

Let $\Omega = \{A_i, {}_iM_j\}_{i,j\in I}$ be a weak simply connected $(D,\mathcal{O})$-species, where each A_i is equal to $\mathcal{O}_i$ or D. One can assign to Ω the quiver $\Gamma(\Omega)$, which is the directed graph defined in Section 1.

The diagram of a $(D,\mathcal{O})$-species Ω is a graph $Q(\Omega)$ that is obtained from $\Gamma(\Omega)$ by deleting the orientation of all arrows which connected unmarked vertices of $\Gamma(\Omega)$. A marked vertex on the diagram will be denoted by $\odot$.

In this section we prove the following theorem:

Theorem 4.1. *Let $\mathcal{O}_i$ be discrete valuation rings with a common skew field of fractions D. Then a weak simply connected $(D,\mathcal{O})$-species Ω is of bounded representation type if and only if the diagram $Q(\Omega)$ is a finite disjoint union of Dynkin diagrams of the forms A_n, D_n, E_6, E_7, E_8 and the following diagrams:*

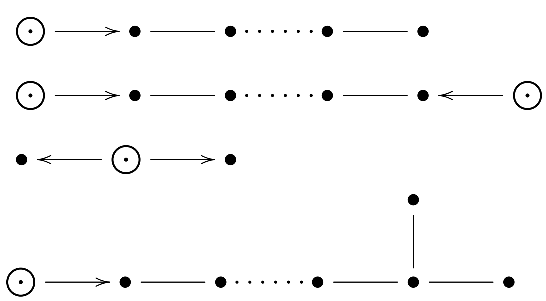
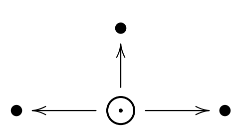


Figure 4.1.

4.1. **Proof of necessity of Theorem 4.1.**

Lemma 4.2. *Let $\mathcal{O}$ be a discrete valuation ring with a skew field of fractions D and the Jacobson radical $R = \pi\mathcal{O} = \mathcal{O}\pi$, $I = \{1,2,3,4\}$. Then a $(D,\mathcal{O})$-species $\Omega = (F_i, {}_iM_j)_{i,j\in I}$ with diagram*



(4.1)

is of unbounded representation type.

Proof. By Section 2 we can reduce the description of representations of $(D,\mathcal{O})$-species with diagram (4.1) to the following matrix problem.

Given a block-rectangular matrix $\mathbf{T} = \begin{array}{|c|}\hline \mathbf{A}_1 \\ \hline \mathbf{A}_2 \\ \hline \mathbf{A}_3 \\ \hline \end{array}$ with the following admissible transformations:

1. Right $\mathcal{O}$-elementary transformations of columns of $\mathbf{T}$.
2. Left D-elementary transformations of rows within each block $\mathbf{A}_i$ $(i = 1,2,3)$.

Set

$$\mathbf{A}_1 = \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{array}\right],$$

$$\mathbf{A}_2 = \left[\begin{array}{cccc|cccc} \pi^{-2} & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \pi^{-4} & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pi^{-2n} & 0 & 0 & \cdots & 1 \end{array}\right], \qquad \mathbf{A}_3 = \left[\begin{array}{c|c} \pi^{n-1} & 0 \\ \pi^{n-2} & 0 \\ \vdots & \vdots \\ 1 & 0 \end{array}\right],$$

where $\pi \in R = \operatorname{rad}\mathcal{O}$, $\pi \neq 0$. By [9, Lemma 3], the matrix $\mathbf{T}$ is indecomposable and therefore the species with diagram (4.4) is of unbounded representation type. $\square$

Lemma 4.3. *Let $\mathcal{O}$ be a discrete valuation ring with a skew field of fractions D and the Jacobson radical $R = \pi\mathcal{O} = \mathcal{O}\pi$, $I = \{1,2,3,4\}$. Then a $(D,\mathcal{O})$-species $\Omega = (F_i, {}_iM_j)_{i,j\in I}$ with diagram*

$$\bullet \longleftarrow \odot \longrightarrow \bullet \longrightarrow \bullet \tag{4.2}$$

is of unbounded representation type.

Proof. We have the following matrix problem.

Given a block-rectangular matrix $\mathbf{T} = \begin{array}{|c|c|}\hline \mathbf{A}_1 & \mathbf{O} \\ \hline \mathbf{A}_2 & \mathbf{A}_3 \\ \hline \end{array}$. The following transformations are admitted:

1. Right $\mathcal{O}$-elementary transformations of columns within the first vertical strip of the matrix $\mathbf{T}$.
2. Right D-elementary transformations of columns within the second vertical strip of the matrix $\mathbf{T}$.
3. Left D-elementary transformations of rows within each horizontal strip of the matrix $\mathbf{T}$.

We reduce $\mathbf{A}_3$ to the form $\begin{array}{|c|c|}\hline \mathbf{I} & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{O} \\ \hline \end{array}$ and obtain the matrix $\mathbf{A}_2 = \begin{array}{|c|}\hline \mathbf{B}_1 \\ \hline \mathbf{B}_2 \\ \hline \end{array}$. It is possible to add any row of $\mathbf{B}_2$ multiplied on the left by an element of D to any row of $\mathbf{B}_1$. Thus the matrices $\mathbf{B}_2$, $\mathbf{B}_1$ and $\mathbf{A}_1$ form the matrix problem II from [9] which is of unbounded representation type by [9, Lemma 4.2]. Therefore the $(D, \mathcal{O})$-species with diagram (4.6) is of unbounded representation type. $\square$

Lemma 4.4. *Let $\mathcal{O}_i$ be a discrete valuation ring with a skew field of fractions D and the Jacobson radical $R_i = \pi_i \mathcal{O}_i = \mathcal{O}_i \pi_i$ for $i = 1, 2$, and let $I = \{1, 2, 3, 4\}$. Then a $(D, \mathcal{O})$-species $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ with diagram*

$$\begin{tikzcd} & \bullet \arrow[d, dash] & \\ \odot \arrow[r] & \bullet & \odot \arrow[l] \end{tikzcd} \tag{4.3}$$

is of unbounded representation type.

Proof. The problem of classifying representations of this species is reduced to the following matrix problem:

Given a block-rectangular matrix $\mathbf{T} = \begin{array}{|c|c|c|}\hline \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \\ \hline \end{array}$. The following transformations are admitted:

1. Left D-elementary transformations of rows of $\mathbf{T}$.
2. Right $\mathcal{O}_1$-elementary transformations of columns of $\mathbf{A}_1$.
3. Right $\mathcal{O}_2$-elementary transformations of columns of $\mathbf{A}_2$.
4. Right D-elementary transformations of columns of $\mathbf{A}_3$.

Consider two possible cases.

Case 1. Assume that $\mathcal{O}_1 = \mathcal{O}_2$. Set

$$\mathbf{A}_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} \pi^2 & 0 & \cdots & 0 \\ 0 & \pi^4 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pi^{2n} \end{pmatrix}, \quad \mathbf{A}_3 = \begin{pmatrix} 1 \\ \pi \\ \vdots \\ \pi^{n-1} \end{pmatrix}.$$

By [9, Lemma 3], the matrix $\mathbf{T}$ is indecomposable. Thus the corresponding representation V of the species Ω is indecomposable and the species Ω with diagram (4.3) is of unbounded representation type.

Case 2. Assume that $\mathcal{O}_1 \neq \mathcal{O}_2$. Set $\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{I}$ and $\mathbf{T} = \begin{array}{|c|c|c|}\hline \mathbf{I} & \mathbf{I} & \mathbf{A}_3 \\ \hline \end{array}$. For the matrix $\mathbf{A}_3$ we obtain the matrix problem II from [10], which is of unbounded representation type by [10, Lemma 4.2]. Therefore, the species with diagram (4.3) is of unbounded representation type. $\square$

The following lemma is proved analogously.

Lemma 4.5. *Let $\mathcal{O}$ be a discrete valuation ring with a skew field of fractions D and the Jacobson radical $R = \pi\mathcal{O} = \mathcal{O}\pi$, and let $I = \{1, 2, 3, 4, 5\}$. Then a weak $(D, \mathcal{O})$-species $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ with diagram*

$$\begin{tikzcd} & \odot \arrow[d] & & \\ \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \end{tikzcd} \tag{4.4}$$

is of unbounded representation type for each orientation of horizontal arrows.

Since we consider only simply connected species, all diagrams that are not presented in diagram 4.2 have a subdiagram of one of the types discussed above in this section and hence they are of unbounded representation type.

Therefore, the necessity of Theorem 4.1 follows from Lemmas 4.2, 4.3, 4.4, and 4.5.

4.2. **Proof of sufficiency of Theorem 4.1.**

Lemma 4.6. *Let $\mathcal{O}$ be a discrete valuation ring with a skew field of fractions D and the Jacobson radical $R = \pi\mathcal{O} = \mathcal{O}\pi$, and let $I = \{1, 2, \ldots, n, n+1\}$. Then a weak $(D, \mathcal{O})$-species $\Omega = (F_i, {}_iM_j)_{i,j \in I}$ with diagram*

$$\begin{tikzcd} \odot \arrow[r] & \bullet \arrow[r, dash] & \bullet \arrow[r, dotted, dash] & \bullet \arrow[r, dash] & \bullet \end{tikzcd} \tag{4.5}$$

is of bounded representation type for any orientation of undirected edges.

Proof. Due to the Gabriel theorem [7], the problem of classifying representations of the species with diagram (4.5) is reduced to the following matrix problem.

Given a block-rectangular matrix $\mathbf{T}$ with entries in a skew field D that is partitioned into $2n-2$ vertical strips:

$$\mathbf{T} = \begin{array}{|c|c|c|c|c|} \hline \mathbf{T}_1 & \mathbf{T}_2 & \cdots & \mathbf{T}_{2n-3} & \mathbf{T}_{2n-2} \\ \hline \end{array}$$

The transformations of the following types are admitted:

1. Right $\mathcal{O}$-elementary transformations of rows of $\mathbf{T}$.
2. Left D-elementary transformations of columns within each vertical strip $\mathbf{T}_i$.
3. Additions of columns of the i-th vertical strip $\mathbf{T}_i$ multiplied on the right by elements of D to columns of the j-th vertical strip $\mathbf{T}_j$ if $i \leq j$.

The matrix $\mathbf{T}$ can be reduced by these transformations to the form in which each block $\mathbf{T}_i$ has the form $\left(\begin{smallmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and all blocks over and under the matrix $\mathbf{I}$ are zero. Thus, for any indecomposable representation, its corresponding matrix has a finite number of elements distinct from zero, and this number depends only on n. Therefore, the species Ω with diagram (4.5) is of bounded representation type. $\square$

Lemma 4.7. *Let $\mathcal{O}_i$ be discrete valuation rings with a common skew field of fractions D and the Jacobson radicals $R_i = \pi_i\mathcal{O}_i = \mathcal{O}_i\pi_i$ for $i = 1, 2$, and let $I = \{1, 2, \dots, n, n+1\}$. Then a weak $(D, \mathcal{O})$-species $\Omega = (F_i, {}_iM_j)_{i,j\in I}$ with diagram*

$$\odot \longrightarrow \bullet \longrightarrow \bullet \cdots\cdots \bullet \longrightarrow \bullet \longleftarrow \odot \tag{4.6}$$

is of bounded representation type for any orientation of undirected edges in the diagram.

Proof. Due to Lemma 4.6, the problem of classifying representations of the species with diagram (4.6) is reduced to the following matrix problem.

Given a block-rectangular matrix $\mathbf{T}$ with entries in a skew field D that is partitioned into n vertical strips:

$$\mathbf{T} = \begin{array}{|c|c|c|c|}\hline \mathbf{T}_1 & \mathbf{T}_2 & \cdots & \mathbf{T}_n \\ \hline \end{array}$$

The transformations of the following types are admitted:

1. Left $\mathcal{O}_2$-elementary transformations of rows of $\mathbf{T}$.
2. Right $\mathcal{O}_1$-elementary transformations of columns within the vertical strip $\mathbf{T}_k$ for some fixed $1 \le k \le n$.
3. Right D-elementary transformations of columns within each vertical strip $\mathbf{T}_i$ if $i \ne k$.
4. Additions of columns in the vertical strip $\mathbf{T}_i$ multiplied on the right by elements of D to columns in the vertical strip $\mathbf{T}_j$ if $i \le j$.

Using these transformations and taking into account [10, Lemma 4.1], one can reduce the matrix $\mathbf{T}$ to the form in which any block $\mathbf{T}_i$ is of the form: $\begin{pmatrix}\mathbf{I} & 0\\ 0 & 0\end{pmatrix}$ or $\begin{pmatrix}\pi^m\mathbf{I} & 0\\ 0 & 0\end{pmatrix}$, and all matrices over, under, on the left, and on the right of the matrix $\mathbf{I}$ (or, respectively, $\pi^m\mathbf{I}$) are zero. Thus, the species with diagram 4.6 is of bounded representation type. $\square$

The following lemmas are proved analogously.

Lemma 4.8. *Let $\mathcal{O}$ be a discrete valuation ring with a skew field of fractions D and the Jacobson radical $R = \pi\mathcal{O} = \mathcal{O}\pi$, and let $I = \{1, 2, \dots, n, n+1\}$. Then a weak $(D, \mathcal{O})$-species $\Omega = (F_i, {}_iM_j)_{i,j\in I}$ with diagram*

$$\begin{tikzcd}[ampersand replacement=\&] \& \& \& \& \bullet \arrow[d, dash] \\ \odot \arrow[r] \& \bullet \arrow[r, dash] \& \bullet \cdots\cdots \bullet \arrow[r, dash] \& \bullet \arrow[r, phantom] \& \bullet \arrow[r, dash] \& \bullet \end{tikzcd} \tag{4.7}$$

is of bounded representation type for any orientation of undirected edges in the diagram.

Lemma 4.9. *Let $\mathcal{O}$ be a discrete valuation ring with a skew field of fractions D and the Jacobson radical $R = \pi\mathcal{O} = \mathcal{O}\pi$, and let $I = \{1, 2, 3\}$. Then a weak $(D, \mathcal{O})$-species $\Omega = (F_i, {}_iM_j)_{i,j\in I}$ with diagram*

$$\bullet \longleftarrow \odot \longrightarrow \bullet \tag{4.8}$$

is of bounded representation type.

The sufficiency of Theorem 4.1 follows from Lemmas 4.6, 4.7, 4.8, 4.9.

5. Right hereditary SPSD-rings of B.R.T.

The structure of right hereditary semiperfect and semidistributive rings (SPSD-rings for short) was described by Kirichenko [14]. In this section we show that the description of right hereditary SPSD-rings of bounded representation type can be reduced to the description of simply connected min-marked $(D, \mathcal{O})$-species of bounded representation type.

Proposition 5.1. *If Ω is a simply connected min-marked $(D, \mathcal{O})$-species of bounded representation type, then the corresponding tensor algebra $\mathfrak{T}(\Omega)$ is a right hereditary SPSD-ring of bounded representation type.*

Proof. Since the species $\Omega = \{F_i, {}_iM_j\}_{i,j\in I}$ is simply connected and each F_i is a prime hereditary Noetherian ring, the tensor algebra $\mathfrak{T}(\Omega)$ is a right Noetherian semiperfect primely triangular ring.

Suppose that ${}_iM_j = 0$ for $i \geq j$. Denote by e_i the identity of the ring F_i. Then $1 = e_1 + e_2 + \cdots + e_n$ is a triangular prime decomposition of the identity of the ring $A = \mathfrak{T}(\Omega)$ and

$$A_{ij} = e_iAe_j = \bigoplus_{(k_1,\ldots,k_s)} {}_iM_{k_1} \otimes_{F_{k_1}} \cdots \otimes_{F_{k_s}} {}_{k_s}M_j,$$

where $A_{ii} = F_i$. Since Ω is a $(D, \mathcal{O})$-species of bounded representation type, the tensor algebra $\mathfrak{T}(\tilde{\Omega})$ is of finite representation type. Therefore eAf is a uniserial left eAe-module and right fAf-module for local idempotents e and f of the ring A. By [11, Theorem 14.2.1] $A = \mathfrak{T}(\Omega)$ is a semidistributive ring.

Since the species Ω is simply connected, one can show that $\mathfrak{T}(\Omega)$ is a right hereditary ring reasoning as on the proof of [3, Theorem 2.6]. $\square$

We show that the inverse statement is also true.

Proposition 5.2. *Let A be a right hereditary SPSD-ring such that its right classical ring of fractions $\tilde{A}$ is a basic ring. If A is of bounded representation* *type, then it is isomorphic to the tensor algebra of a simply connected weak $(D,\mathcal{O})$-species of bounded representation type.*

Proof. Let A be a right hereditary SPSD-ring such that its right classical ring of fractions $\tilde{A}$ is a basic ring. Then by [14, Corollary 3.12] A is a primely triangular ring with triangular prime decomposition of the identity $1 = e_1 + e_2 + \cdots + e_n$ such that $e_iAe_i = \mathcal{O}_i$ for $i = 1,\dots,k$ and $e_jAe_j = D$ for $j = k+1,\dots,n$. Since A is a right hereditary primely triangular ring, A_{ij} is a left $\tilde{F}_i$-module. Then the $(D,\mathcal{O})$-species $\Omega_A = (F_i, {}_iM_j)_{i,j\in I}$ can be assigned by the ring A by setting $I = \{1,2,\dots,n\}$, $F_i = e_iAe_i$ and ${}_iM_j = A_{ij}/\sum_{i<k<j} A_{ik}A_{kj}$, where $A_{ij} = e_iAe_j$ is an $(\tilde{F}_i,\tilde{F}_j)$-bimodule. Since ${}_iM_i = 0$, ${}_iM_j = 0$ if $F_j = \mathcal{O}_j$, and $\mathrm{Hom}_A(e_iA, e_jA) \neq 0$ if $\mathrm{Hom}_A(e_jA, e_iA) = 0$, the quiver $\Gamma(\Omega_A)$ is without loops and oriented cycles, and minimal marked vertices of $\Gamma(\Omega_A)$ are points $1,2,\dots,k$, i.e. Ω_A is a weak simply connected species. It is easy to show that the quiver $\Gamma(\Omega_A)$ is in fact the prime quiver of the right hereditary SPSD-ring A.

Since A is a right hereditary SPSD-ring, its prime quiver contains no circuits by [14, Theorem 3.8]. By [6, Proposition 3], A is a split ring. Thus, if $\mathfrak{T}(\Omega_A)$ is a tensor algebra corresponding to the $(D,\mathcal{O})$-species Ω_A, then A is isomorphic to $\mathfrak{T}(\Omega_A)$ by [6, Theorem 3]. $\square$

Let A be a right hereditary SPSD-ring such that its right classical ring of fractions $\tilde{A}$ is a basic ring. Then the prime quiver $PQ(A)$ of A is a simply connected quiver that coincides with the diagram of some finite poset S with weights $\mathcal{O}_i$ or D, where all $\mathcal{O}_i$ are discrete valuation rings with a common classical division ring of fractions D, and all points with weights $\mathcal{O}_i$ correspond to the minimal elements of S. Let $\overline{PQ(A)}$ be a graph obtained from the diagram $PQ(A)$ by deleting the orientation of all arrows except those that connect points with weights $\mathcal{O}_i$.

Proposition 5.2 and Theorem 4.1 ensure the following theorem.

Theorem 5.3. *Let A be a right hereditary SPSD-ring whose classical ring of fractions $\tilde{A}$ is a basic ring. Then A is of bounded representation type if and only if the diagram $\overline{PQ(A)}$ of the prime quiver $PQ(A)$ is a finite disjoint union of Dynkin diagrams of the forms A_n, D_n, E_6, E_7, E_8 and the diagrams with weights from Figure* 4.2.

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