GENERALIZED INJECTIVITY OF BANACH MODULES

MOHAMMAD FOZOUNI

Abstract. In this paper, we study the notion of φ-injectivity in the special case that φ = 0. For an arbitrary locally compact group G, we characterize the 0-injectivity of \( L^1(G) \) as a left \( L^1(G) \) module. Also, we show that \( L^1(G)^{**} \) and \( L^p(G) \) for \( 1 < p < \infty \) are 0-injective Banach \( L^1(G) \) modules.

1. Introduction

The homological properties of Banach modules such as injectivity, projectivity, and flatness were first introduced and investigated by Helemskii; see [5, 6]. White in [11] gave a quantitative version of these concepts, i.e., he introduced the concepts of \( C \)-injective, \( C \)-projective, and \( C \)-flat Banach modules for a positive real number \( C \). Recently Nasr-Isfahani and Soltani Renani introduced a version of these homological concepts based on a character of a Banach algebra \( A \) and they showed that every injective (projective, flat) Banach module is a character injective (character projective, character flat respectively) module but that the converse is not valid in general. With the use of these new homological concepts, they gave a new characterization of \( \phi \)-amenability of a Banach algebra \( A \) such that \( \phi \in \Delta(A) \) and a necessary condition for \( \phi \)-contractibility of \( A \); see [8].

2. Preliminaries

Let \( A \) be a Banach algebra and \( \Delta(A) \) denote the character space of \( A \), i.e., the space of all non-zero homomorphisms from \( A \) onto \( \mathbb{C} \). We denote by \( A\text{-mod} \) and \( \text{mod-}A \) the category of all Banach left \( A \)-modules and all Banach right \( A \)-modules, respectively. In the case that \( A \) has an identity we denote by \( A\text{-unmod} \) the category of all Banach left unital modules. For \( E, F \in A\text{-mod} \), let \( A B(E, F) \) be the space of all bounded linear left \( A \)-module morphisms from \( E \) into \( F \).

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For each Banach space $E$, $B(A, E)$; the Banach space consisting of all bounded linear operator from $A$ into $E$, is in $\textbf{A-mod}$ with the following module action:

$$(a \cdot T)(b) = T(ba) \quad (T \in B(A, E), a, b \in A).$$

**Definition 2.1.** Let $A$ be a Banach algebra and $J \in \textbf{A-mod}$. We say that $J$ is injective if for each $F, E \in \textbf{A-mod}$ and admissible monomorphism $T : F \to E$ the induced map $T_J : _AB(E, J) \to _AB(F, J)$ defined by $T_J(R) = R \circ T$ is onto.

Suppose that $\phi \in \Delta(A)$. For $E \in \textbf{A-mod}$, put

$$I(\phi, E) = \text{span}\{a \cdot \xi - \phi(\xi)a : a \in A, \xi \in E\},$$

$$\phi B(A^2, E) = \{T \in B(A^2, E) : T(ab - \phi(b)a) = a \cdot T(b - \phi(b)e^2), (a, b \in A)\}.$$ 

It is clear that $I(\phi, E) = \{0\}$ if and only if the module action of $E$ is given by $a \cdot x = \phi(a)x$ for all $a \in A$ and $x \in E$.

Obviously, $\phi B(A^2, E)$ is a Banach subspace of $B(A^2, E)$. On the other hand, for each $b \in \ker(\phi)$, if $T \in \phi B(A^2, E)$, then $T(ab) = a \cdot T(b)$ for all $a \in A$. Therefore, we conclude that $\phi B(A^2, E)$ is a Banach left $A$-submodule of $B(A^2, E)$.

Note that if $E, F \in \textbf{A-mod}$ and $\rho : E \to F$ is a left $A$-module homomorphism, we can extend the module actions of $E$ and $F$ from $A$ into $A^2$ and $\rho$ to a left $A^2$-module homomorphism in the following way:

$$(a, \lambda) \cdot e = a \cdot e + \lambda e \quad (a \in A, \lambda \in \mathbb{C}, e \in E)$$

$$(a, \lambda) \cdot f = a \cdot f + \lambda f \quad (a \in A, \lambda \in \mathbb{C}, f \in F).$$

So, $\rho((a, \lambda) \cdot e) = a \cdot \rho(e) + \lambda \rho(e) = (a, \lambda) \cdot \rho(e)$.

For Banach spaces $E$ and $F$, $T \in B(E, F)$ is admissible if and only if there exists $S \in B(F, E)$ such that $T \circ S \circ T = T$.

The following definition of a $\phi$-injective Banach module, was introduced by Nasr-Isfahani and Soltani Renani in [8].

**Definition 2.2.** Let $A$ be a Banach algebra, $\phi \in \Delta(A)$ and $J \in \textbf{A-mod}$. We say that $J$ is $\phi$-injective if for each $F, E \in \textbf{A-mod}$ and admissible monomorphism $T : F \to E$ with $I(\phi, E) \subseteq \text{Im}T$, the induced map $T_J$ is onto.

By Definitions 2.1 and 2.2, one can easily check that each injective module is $\phi$-injective, although by [8, Example 2.5], the converse is not valid. In [4], the authors with the use of the semigroup algebras, gave two good examples of $\phi$-injective Banach modules which are not injective.

Let $E, F$ be in $\textbf{A-mod}$. An operator $T \in _AB(E, F)$ is called a retraction if there exists an $S \in _AB(F, E)$ such that $T \circ S = \text{Id}_F$. In this case...
$F$ is called a retract of $E$. Also, an operator $T \in \mathcal{A}B(E, F)$ is called a coretraction if there exists an $S \in \mathcal{A}B(F, E)$ such that $S \circ T = \text{Id}_E$.

For $E \in \mathcal{A}\text{-mod}$, let $\phi \Pi^2 : E \to \phi B(A^\sharp, E)$ be defined by $\phi \Pi^2(x)(a) = a \cdot x$ for all $a \in A^\sharp$ and $x \in E$.

**Theorem 2.3.** [8, Theorem 2.4] Let $A$ be a Banach algebra and $\phi \in \Delta(A)$. For $J \in \mathcal{A}\text{-mod}$ the following statements are equivalent.

1. $J$ is $\phi$-injective.
2. $\phi \Pi^2 \in \mathcal{A}B(J, \phi B(A^\sharp, J))$ is a coretraction.

### 3. 0-injectivity of Banach modules

In this section, we give the definition of a 0-injective Banach left $A$-module and show that this class of Banach modules are strictly larger than the class of injective Banach modules.

For each $E \in \mathcal{A}\text{-mod}$ define

$$0B(A^\sharp, E) = \{T \in B(A^\sharp, E) : T(ab) = a \cdot T(b) \text{ for all } a, b \in A\}.$$  

Clearly, $0B(A^\sharp, E)$ is a Banach left $A$-submodule of $B(A^\sharp, E)$. It is well-known that $E^*$ is in $\mathcal{A}\text{-mod}$ with the following module action:

$$(f \cdot a)(x) = f(a \cdot x) \quad (a \in A, x \in E, f \in E^*).$$

**Definition 3.1.** Let $A$ be a Banach algebra and $E \in \mathcal{A}\text{-mod}$. We say that $E$ is a (left) 0-injective if for each $F, K \in \mathcal{A}\text{-mod}$ and admissible monomorphism $T : F \to K$ for which $A \cdot K = \text{span}\{a \cdot k : a \in A, k \in K\} \subseteq \text{Im}T$, the induced map $T_J$ is onto.

Similarly, one can define the concept of (right) 0-injective $A$-module. We say that $E \in \mathcal{A}\text{-mod}$ is 0-flat if $E^* \in \mathcal{A}\text{-mod}$ is (right) 0-injective.

Clearly, each injective module is 0-injective.

We use of the following characterization of 0-injectivity in the sequel without giving a reference.

**Proposition 3.2.** Let $A$ be a Banach algebra and $E \in \mathcal{A}\text{-mod}$. Then $E$ is 0-injective if and only if $0 \Pi^2$ is a coretraction.

**Proof.** Suppose $E \in \mathcal{A}\text{-mod}$ is 0-injective. Take $F = E$, $K = 0B(A^\sharp, E)$ and $T = 0 \Pi$. Then $A \cdot K \subseteq \text{Im}(0 \Pi)$ and $a \cdot T = 0 \Pi(T(a))$ for each $a \in A$ and $T \in K$. Hence, for the identity map $I_E \in A B(F, E) = A B(E, E)$, there exists $\rho \in A B(K, E) = A B(0B(A^\sharp, E), E)$ such that $\rho \circ 0 \Pi = \rho \circ T = I_E$.

Conversely, let $0 \rho : 0B(A^\sharp, E) \to E$ be a left $A$-module morphism and a left inverse for the canonical morphism $0 \Pi$. Suppose that $F, K \in \mathcal{A}\text{-mod}$ and $T : F \to K$ is an admissible monomorphism such that $A \cdot K \subseteq \text{Im}T$. Let $W \in A B(F, E)$ and define the map $R : K \to 0B(A^\sharp, E)$ by

$$R(k)(a) = W \circ T'(a \cdot k) \quad (k \in K, a \in A^\sharp),$$
where $T' \in B(K, F)$ satisfies $T \circ T' \circ T = T$. We show that $R$ is well defined, i.e., $R(k) \in \mathcal{A}A^\sharp, E)$ for each $k \in K$. So, we will show that $R(k)(ab) = a \cdot R(k)(b)$ for each $a, b \in A$. By assumption $A \cdot K \subseteq \text{Im}T$ and so there exist $f \in F$ such that $b \cdot k = T(f)$. Therefore

$$a \cdot R(k)(b) = a \cdot W \circ T'(b \cdot k) = a \cdot W \circ T'(T(f)) = a \cdot W(f) = W(a \cdot f) = W \circ T'(a \cdot f) = W \circ T'(ab \cdot k) = R(k)(ab).$$

Moreover, for each $b \in A^\sharp$ we have

$$R(a \cdot k)(b) = W \circ T'(b \cdot (a \cdot k)) = W \circ T'(ba \cdot k) = R(k)(ba) = (a \cdot R(K))(b).$$

It follows that $R(a \cdot k) = a \cdot R(k)$. Now, take $S = \rho \circ R \in A B(K, E)$. Since $R \circ T = \rho \circ W$, we conclude that $S \circ T = W$, which completes the proof. \qed

Now, we give a sufficient condition for 0-injectivity which provides us a large class of Banach algebras $A$ such that they are 0-injective in $\mathbf{A-mod}$.

Recall that by [10, Corollary 2.2.8(i)], if $A \in \mathbf{A-mod}$ is injective, then $A$ has a right identity. Moreover, the converse is not valid in general even in the case that $A$ has an identity; see Example 3.4.

**Proposition 3.3.** Let $A$ be a Banach algebra. If $A$ has an identity, then $A \in \mathbf{A-mod}$ is 0-injective.

**Proof.** Let $e$ be the identity of $A$. Define $\rho : \mathcal{A}A^\sharp, A) \to A$ by $\rho(T) = T(e)$ for all $T \in \mathcal{A}A^\sharp, A)$. It is obvious that $\rho$ is a left inverse for $\mathcal{A}A^\sharp, \mathcal{A})$, because for each $a \in A$, we have

$$\rho \circ \mathcal{A}A^\sharp(a) = (\mathcal{A}A^\sharp(a))(e) = ea = a.$$

Also, $\rho$ is a left $A$-module morphism, because for each $a \in A$ and $T \in \mathcal{A}A^\sharp, A$ we have

$$\rho(a \cdot T) = (a \cdot T)(e) = T(ea) = T(a)$$

$$a \cdot \rho(T) = a \cdot T(e) = T(ae) = T(a).$$

Therefore, $A \in \mathbf{A-mod}$ is 0-injective. \qed

For each locally compact group $G$, let $M(G)$ be the Banach algebra consisting of all complex regular Borel measure of $G$ and let $L^\infty(G)$ be the space of all measurable complex-valued functions on $G$ which are essentially bounded; see [1] for more details.
The group $G$ is said to be amenable if there exists an $m \in L^\infty(G)^*$ such that $m \geq 0$, $m(1) = 1$ and $m(L_x f) = m(f)$ for each $x \in G$ and $f \in L^\infty(G)$, where $L_x f(y) = f(x^{-1}y)$.

Regarding the last proposition we give the following example which shows the difference between 0-injectivity and injectivity.

**Example 3.4.** Let $G$ be a non-amenable locally compact group. Then by [10, Theorem 3.1.2], $M(G) \in \mathcal{M}(G)$-mod is not injective, but it is 0-injective.

By [6, Proposition VII.1.35], if $E \in \mathbf{A}$-unmod, each retract of $E$ is injective. For 0-injective Banach modules we have the following proposition.

**Proposition 3.5.** Let $A$ be a Banach algebra and let $E \in \mathbf{A}$-mod be 0-injective. Then each retract of $E$ is also 0-injective.

**Proof.** Let $F \in \mathbf{A}$-mod be a retract of $E$. Also, let $T \in A(B(E, F)$ and $S \in A(B(F, E)$ be such that $T \circ S = I_F$.

Since $E \in \mathbf{A}$-mod is 0-injective, there exists $E \rho_e \in A(B(\mathbf{A}, E), E)$ for which $E \rho_e \circ E \Pi_e(x) = x$ for all $x \in E$.

Now, define the map $F \rho_e : \mathbf{A}(\mathbf{A}, F) \to F$ by

$$F \rho_e(W) = T \circ E \rho_e(S \circ W) \quad (W \in \mathbf{A}(\mathbf{A}, F)).$$

It is straightforward to check that $F \rho_e$ is a left $A$-module morphism. On the other hand, for each $y \in F$ we have

$$F \rho_e \circ F \Pi_e(y) = F \rho_e(F \Pi_e(y)) = T \circ E \rho_e(S \circ F \Pi_e(y)) = T \circ E \rho_e(E \Pi_e(S(y))) = T \circ S(y) = y.$$

Therefore, $F \in \mathbf{A}$-mod is 0-injective. \hfill \Box

Now, we try to characterize 0-injectivity of $L^1(G)$ in $L^1(G)$-mod. First we give the following lemma.

**Lemma 3.6.** Let $A$ be a Banach algebra and $E \in \mathbf{A}$-mod. If $E$ is 0-injective, then

$$0B(A, E) = \{ T : T = R_x \text{ on } A \text{ for some } x \in E \},$$

where $R_x a = a \cdot x$ for all $a \in A$.

**Proof.** Let $E \in \mathbf{A}$-mod be 0-injective. So, there exists $0 \rho_e \in A(B(\mathbf{A}, E), E)$ with $0 \rho_e \circ 0 \Pi_e(x) = x$ for all $x \in E$.

Let $T$ be an element of $0B(A, E)$. Hence

$$T(b) = 0 \rho_e \circ 0 \Pi_e(T(b)) = 0 \rho_e(0 \Pi_e(T(b))) = 0 \rho_e(b \cdot T) = b \cdot 0 \rho_e(T).$$
Take \( x_0 = _0\rho^\sharp(T) \). So, \( T = R_{x_0} \) on \( A \) and this completes the proof. \( \square \)

Recall that \( E \in \mathbf{A-mod} \) is faithful in \( A \), if for each \( x \in E \), the relation \( a \cdot x = 0 \) for all \( a \in A \), implies \( x = 0 \).

**Theorem 3.7.** Let \( G \) be a locally compact group. Then \( L^1(G) \in \mathbf{L^1(G)-mod} \) is 0-injective if and only if \( G \) is discrete.

**Proof.** Let \( G \) be a discrete group. Then \( L^1(G) \) is unital and so the result follows from Proposition 3.3.

Conversely, let \( G \) be non-discrete. So, \( L^1(G) \neq M(G) \). Suppose that \( \mu \in M(G) \setminus L^1(G) \). Since \( L^1(G) \) is an ideal of \( M(G) \), the operator \( T_\mu \) defined by

\[
T_\mu((f,\lambda)) = f \cdot \mu \quad ((f,\lambda) \in L^1(G)\!^\sharp),
\]

is in \( _0B(L^1(G)\!^\sharp, L^1(G)) \), but it is not of the form \( R_x \) for some \( x \in L^1(G) \), because \( M(G) \) is faithful in \( L^1(G) \). Therefore, by Lemma 3.6, \( L^1(G) \) in \( \mathbf{L^1(G)-mod} \) is not 0-injective. \( \square \)

Recall that a Banach algebra \( A \) is left 0-amenable if for every Banach \( A \)-bimodule \( X \) with \( a \cdot x = 0 \) for all \( a \in A \) and \( x \in X \), every continuous derivation \( D : A \to X^* \) is inner, or equivalently, \( H^1(A,X^*) = 0 \) where \( H^1(A,X^*) \) denotes the first cohomology group of \( A \) with coefficients in \( X^* \); see [7] for more details.

Now, we investigate the relation between 0-injectivity and 0-amenability.

Let \( E,F \in \mathbf{A-mod} \). Suppose that \( Z^1(A \times E,F) \) denotes the Banach space of all continuous bilinear maps \( B : A \times E \to F \) satisfying

\[
a \cdot B(b,\xi) - B(ab,\xi) + B(a,b \cdot \xi) = 0 \quad (a,b \in A, \xi \in E).
\]

Define \( \delta_0 : B(E,F) \to Z^1(A \times E,F) \) by \( (\delta_0T)(a,\xi) = a \cdot T(\xi) - T(a \cdot \xi) \) for all \( a \in A \) and \( \xi \in E \). Then we have

\[
\text{Ext}^1_A(E,F) = Z^1(A \times E,F)/\text{Im}\delta_0.
\]

By [5, Proposition VII.3.19], we know that \( \text{Ext}^1_A(E,F) \) is topologically isomorphic to \( H^1(A,B(E,F)) \) where \( B(E,F) \) is a Banach \( A \)-bimodule with the following module actions:

\[
(a \cdot T)(\xi) = a \cdot T(\xi), \quad (T \cdot a)(\xi) = T(a \cdot \xi) \quad (a \in A, \xi \in E, T \in B(E,F)).
\]

To see further details about \( \text{Ext}^1_A(E,F) \); see [6].

**Lemma 3.8.** Let \( E \in \mathbf{A-mod} \). If \( \text{Ext}^1_A(F,E) = \{0\} \) for all \( F \in \mathbf{A-mod} \) with \( A \cdot F = 0 \), then \( E \in \mathbf{A-mod} \) is 0-injective.

**Proof.** To show this, let \( K,W \in \mathbf{A-mod} \) and \( T : K \to W \) be an admissible monomorphism with \( A \cdot W \subseteq \text{Im}T \). We claim that the induced map \( T_E \) is onto.
We know that the short complex \( 0 \to K \xrightarrow{T} W \xrightarrow{q} \frac{W}{\text{Im}T} \to 0 \) is admissible where \( q \) is the quotient map. But for all \( a \in A \) and \( x \in W \), \( a \cdot (x + \text{Im}T) = \text{Im}T \), because \( A \cdot W \subseteq \text{Im}T \). Therefore, by assumption \( \text{Ext}^1_A(\frac{W}{\text{Im}T}, E) = \{0\} \). Now, by [6, III Theorem 4.4], the complex
\[
0 \to \mathcal{A}(\frac{W}{\text{Im}T}, E) \to \mathcal{A}(W, E) \xrightarrow{T_E} \mathcal{A}(K, E) \to \text{Ext}^1_A(\frac{W}{\text{Im}T}, E) \to \cdots,
\]
is exact. Therefore, \( T_E \) is onto.

Recall that if \( E, F \) be two Banach spaces and \( E \hat{\otimes} F \) denotes the projective tensor product space, then \( (E \hat{\otimes} F)^* \) is isomorphic to \( B(E, F^*) \) as two Banach spaces with the pairing
\[
<Tx, y> = T(x \otimes y) \quad (x \in E, y \in F, T \in (E \hat{\otimes} F)^*).
\]
Also, note that \( E \hat{\otimes} F \) is isometrically isomorphic to \( F \hat{\otimes} E \) as two Banach spaces.

**Theorem 3.9.** Let \( A \) be a Banach algebra. Then \( A \) is left 0-amenable if and only if each \( J \in \text{mod-A} \) is 0-flat.

**Proof.** Suppose that \( A \) is left 0-amenable. We show that \( \text{Ext}^1_A(E, J^*) = \{0\} \) for all \( E \in \text{A-mod} \) with \( A \cdot E = 0 \). We have
\[
\text{Ext}^1_A(E, J^*) = H^1(A, B(E, J^*)) = H^1(A, (E \hat{\otimes} J)^*) = \{0\},
\]
because \( E \hat{\otimes} J \in \text{mod-A} \) has the module action, \( a \cdot z = 0 \) for all \( z \in E \hat{\otimes} J \). Therefore, by Lemma 3.8, \( J^* \in \text{A-mod} \) is 0-injective.

Conversely, let \( J \in \text{mod-A} \) be 0-flat. So, for Banach right \( A \)-module \( C \) with module action \( \lambda \cdot a = 0 \) for all \( a \in A \) and \( \lambda \in C \) we have
\[
H^1(A, J^*) = H^1(A, B(J, C)) = H^1(A, B(J, C^*)) = H^1(A, (J \hat{\otimes} C)^*) = H^1(A, (C \hat{\otimes} J)^*),
\]
\[
= H^1(A, B(C, J^*))
\]
\[
= \text{Ext}^1_A(C, J^*) = 0.
\]
Hence, if we take \( J \) a left \( A \) module with module action \( a \cdot x = 0 \) for all \( a \in A \) and \( x \in J \), then the above relation implies that \( A \) is 0-amenable.

By [2, Corollary 4.7], we know that \( L^1(G)^{**} \in L^1(G)\text{-mod} \) is injective if and only if \( G \) is an amenable group. Also, if \( 1 < p < \infty \) by [3, Theorem 9.6], \( L^p(G) \in L^1(G)\text{-mod} \) is injective if and only if \( G \) is an amenable group.
Corollary 3.10. Let $G$ be a locally compact group, $1 < p < \infty$ and $E \in L^1(G)\text{-mod}$ be $L^p(G)$ or $L^1(G)^{**}$. Then $E \in L^1(G)\text{-mod}$ is $0$-injective.

Proof. Since $L^1(G)$ has a bounded approximate identity by [7, Proposition 3.4 (i)], we know that $L^1(G)$ is $0$-amenable. So, by Theorem 3.9 we conclude the result. The second part follows similarly, because for each $1 < p < \infty$ we know that $L^q(G)^* = L^p(G)$ where $q$ satisfies the relation $q^{-1} + p^{-1} = 1$. \hfill \square

Remark 3.11. In general, by [7, Proposition 3.4 (i)], if $A$ is a Banach algebra with a bounded approximate identity, then each $E \in \text{mod-A}$ is $0$-flat.

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References


(Received: February 25, 2015) Department of Mathematics
(Revised: March 31, 2015) Faculty of Basic Science and Engineering
Gonbad Kavous University, P.O. Box 163
Gonbad-e Kavous, Golestan
Iran
fozouni@gonbad.ac.ir; fozouni@hotmail.com