ON THE CONVOLUTION AND NEUTRIX CONVOLUTION OF THE FUNCTIONS \( \sinh^{-1} x \) AND \( x^r \)

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Abstract. The neutrix convolution \( \sinh^{-1} x \odot x^r \) is evaluated for \( r = 0, 1, 2, \ldots \). Further results are also given.

1. Introduction

The functions \( \sinh^{-1} x_+ \) and \( \sinh^{-1} x_- \) are defined by
\[
\sinh^{-1} x_+ = H(x) \sinh^{-1} x, \quad \sinh^{-1} x_- = H(-x) \sinh^{-1} x,
\]
where \( H \) denotes Heaviside’s function. Note that
\[
\sinh^{-1} x = \sinh^{-1} x_+ + \sinh^{-1} x_-.
\]

If \( f \) and \( g \) are locally summable functions then the classical definition for the convolution \( f \ast g \) of \( f \) and \( g \) is as follows:

Definition 1. Let \( f \) and \( g \) be functions. Then the convolution \( f \ast g \) is defined by
\[
(f \ast g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt
\]
for all points \( x \) for which the integral exists.

It follows easily from the definition that if the classical convolution \( f \ast g \) of \( f \) and \( g \) exists, then \( g \ast f \) exists and
\[
f \ast g = g \ast f.
\]
Further, if \( (f \ast g)' \) and \( f' \ast g \) (or \( f' \ast g \)) exist, then
\[
(f \ast g)' = f' \ast g \quad \text{or} \quad f' \ast g.
\]

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The classical definition of the convolution can be extended to define the convolution \( f * g \) of two distributions \( f \) and \( g \) in \( D' \) with the following definition, see [9].

**Definition 2.** Let \( f \) and \( g \) be distributions in \( D' \). Then the convolution \( f * g \) is defined by the equation

\[
\langle (f * g)(x), \varphi(x) \rangle = \langle f(y), \langle g(x), \varphi(x + y) \rangle \rangle
\]

for arbitrary \( \varphi \) in \( D' \), provided that \( f \) and \( g \) satisfy either of the following conditions:

(a) either \( f \) or \( g \) has bounded support,
(b) the supports of \( f \) and \( g \) are bounded on the same side.

It follows that if the convolution \( f * g \) exists by this definition, then equations (2) and (3) are satisfied.

The following theorems were proved in [10].

**Theorem 1.** The neutrix convolutions \( (\tan^{-1}_+ x) \odot x^{2r+1} \) and \( (\tan^{-1}_+ x) \odot x^{2r} \) exist and

\[
(\tan^{-1}_+ x) \odot x^{2r+1} = \sum_{k=0}^{r} \left( \frac{2r + 1}{2k} \right) (-1)^{k+1} \pi x^{2r-2k+1} \pi x^{2r-2k},
\]

\[
(\tan^{-1}_+ x) \odot x^{2r} = \sum_{k=0}^{r} \left( \frac{2r}{2k} \right) (-1)^{k+1} \pi x^{2r-2k} + \sum_{k=1}^{r} \left( \frac{2r}{2k-1} \right) (-1)^{k} \pi x^{2r-2k+1},
\]

for \( r = 0, 1, 2, \ldots \).

**Theorem 2.** The neutrix convolutions \( x^{2r+1} \odot \tan^{-1}_+ x \) and \( x^{2r} \odot \tan^{-1}_+ x \) exist and

\[
x^{2r+1} \odot \tan^{-1}_+ x = \sum_{k=0}^{r} \left( \frac{2r + 1}{2k} \right) x^{2r-2k+1} G_k(x) - \sum_{k=0}^{r} \left( \frac{2r + 1}{2k+1} \right) x^{2r-2k} F_k(x),
\]

\[
x^{2r} \odot \tan^{-1}_+ x = \sum_{k=0}^{r} \left( \frac{2r}{2k} \right) x^{2r-2k} G_k(x) - \sum_{k=0}^{r-1} \left( \frac{2r}{2k+1} \right) x^{2r-2k-1} F_k(x),
\]

for \( r = 0, 1, 2, \ldots \).

The next theorem was proved in [6].
Theorem 3. If \( \lambda, \lambda + \mu < 0 \) and \( \mu \neq 0 \), then the neutrix convolution \( \text{ei}_-(\lambda x) \ast e^{\mu x} \) exists and

\[
\text{ei}_-(\lambda x) \ast e^{\mu x} = -\mu^{-1} \ln(1 + \mu/\lambda)e^{\mu x}.
\]

The dilogarithm integral \( \text{Li}(x) \) see [6] is defined for by

\[
\text{Li}(x) = -\int_0^x \frac{\ln|1-t|}{t} dt
\]

and the associated functions \( \text{Li}_+(x) \) and \( \text{Li}_-(x) \) are defined by

\[
\text{Li}_+(x) = H(x) \text{Li}(x), \quad \text{Li}_-(x) = H(-x) \text{Li}(x) = \text{Li}(x) - \text{Li}_+(x),
\]

where \( H(x) \) denotes Heaviside’s function.

The following theorem was proved in [6].

Theorem 4. The neutrix convolution \( \text{Li}_+(x) \ast x^r \) exists and

\[
\text{Li}_+(x) \ast x^r = \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} \frac{(-1)^{r-i}}{(r-i+1)^2} x^i
\]

for \( r = 0, 1, 2, \ldots \). In particular

\[
\text{Li}_+(x) \ast H(x) = 1,
\]

\[
\text{Li}_+(x) \ast x_+ = x - \frac{1}{8}.
\]

2. Main results

We need the following lemmas to prove our results on the convolution and neutrix convolution.

Lemma 1.

\[
\sinh^{2r} x = 2^{1-2r} \sum_{k=1}^{r} \left( \frac{2r}{r-k} \right) (-1)^{r-k} \cosh(2kx) + (-1)^r 2^{-2r} \left( \frac{2r}{r} \right), \quad (5)
\]

\[
\sinh^{2r-1} x = 2^{2-2r} \sum_{k=1}^{r} \left( \frac{2r-1}{r-k} \right) (-1)^{r-k} \sinh(2k-1)x, \quad (6)
\]

for \( r = 1, 2, \ldots \).
Proof. We have

\[
\sinh^{2r} x = 2^{-2r} (e^x - e^{-x})^{2r} = 2^{-2r} \sum_{k=0}^{2r} \binom{2r}{k} (-1)^k e^{(2r-2k)x}
\]

\[
= 2^{-2r} \sum_{k=0}^{r-1} \binom{2r}{k} (-1)^k (e^{(2r-2k)x} + e^{-(2r-2k)x}) + (-1)^r 2^{-2r} \binom{2r}{r}
\]

\[
= 2^{1-2r} \sum_{k=1}^{r} \left( \frac{2r}{r-k} \right) (-1)^{r-k} \cosh(2kx) + (-1)^r 2^{-2r} \binom{2r}{r},
\]

proving equation (5).

Similarly, we have

\[
\sinh^{2r-1} x = 2^{1-2r} (e^x - e^{-x})^{2r-1} = 2^{1-2r} \sum_{k=0}^{2r-1} \binom{2r-1}{k} (-1)^k e^{(2r-2k-1)x}
\]

\[
= 2^{1-2r} \sum_{k=0}^{r-1} \binom{2r-1}{k} (-1)^k (e^{(2r-2k-1)x} - e^{-(2r-2k-1)x})
\]

\[
= 2^{2-2r} \sum_{k=1}^{r} \left( \frac{2r-1}{r-k} \right) (-1)^{r-k} \sinh(2k-1)x,
\]

proving equation (6).

For shortness, we will write

\[
\sinh^{r} x = \sum_{k=0}^{r} [a_{r,k} \cosh(kx) + b_{r,k} \sinh(kx)],
\]

(7)

for \( r = 1, 2, \ldots \), where

\[
a_{2r, 2k-1} = 0; \quad k = 1, 2, \ldots, r,
\]

\[
a_{2r-1, k} = 0; \quad k = 0, 1, 2, \ldots, 2r - 1,
\]

\[
b_{2r-1, 2k} = 0; \quad k = 0, 1, 2, \ldots, r - 1,
\]

\[
b_{2r, k} = 0; \quad k = 1, 2, \ldots, 2r
\]

so that

\[
\sinh^{2r} x = \sum_{k=0}^{r} a_{2r, 2k} \cosh(2kx),
\]

(8)

\[
\sinh^{2r-1} x = \sum_{k=1}^{r} a_{2r-1, 2k-1} \sinh(2k-1)x,
\]

(9)

for \( r = 1, 2, \ldots \) and \( k = 1, 2, \ldots, r \). □
Lemma 2.

\[
\sinh(2rx) = \sum_{k=1}^{r} \sum_{j=0}^{k-1} \binom{2r}{2k-1} \binom{k-1}{j} (-1)^{k+j+1} \cosh^{2r-2k+2j+1} x \sinh x,
\]

\[
\sinh(2r-1)x = \sum_{k=1}^{r} \sum_{j=0}^{k-1} \binom{2r-1}{2k-1} \binom{k-1}{j} (-1)^{k+j+1} \cosh^{2r-2k+2j} x \sinh x,
\]

for \( r = 1, 2, \ldots \).

Proof. Using de Moivre’s Theorem, we have

\[
\cos(2rx) + i \sin(2rx) = (\cos x + i \sin x)^{2r}.
\]

Equating the imaginary parts, we have

\[
\sin(2rx) = \sum_{k=1}^{r} \binom{2r}{2k-1} (-1)^{k+1} \cosh^{2r-2k+1} x \sin^{2k-1} x
\]

\[
= \sum_{k=1}^{r} \binom{2r}{2k-1} (-1)^{k+1} \cosh^{2r-2k+1} x (1 - \cos^2 x)^{k-1} \sin x
\]

\[
= \sum_{k=1}^{r} \binom{2r}{2k-1} (-1)^{k+1} \cosh^{2r-2k+1} x \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^{j} \cos^{2j} x \sin x.
\]

(12)

Replacing \( x \) by \( ix \) in equation (12), we get equation (10).

Similarly, we have

\[
\sin(2r-1)x = \sum_{k=1}^{r} \binom{2r-1}{2k-1} (-1)^{k+1} \cosh^{2r-2k} x \sin^{2k-1} x
\]

\[
= \sum_{k=1}^{r} \binom{2r-1}{2k-1} (-1)^{k+1} \cosh^{2r-2k} x (1 - \cos^2 x)^{k-1} \sin x
\]

\[
= \sum_{k=1}^{r} \binom{2r-1}{2k-1} (-1)^{k+j+1} \cosh^{2r-2k} x \sum_{j=0}^{k-1} \binom{k-1}{j} \cos^{2j} x \sin x.
\]

(13)

Replacing \( x \) by \( ix \) in equation (13), we get equation (11).

For shortness, we will write

\[
\sinh(rx) = \sum_{k=1}^{r} c_{r,k} \cosh^{k} x \sinh x,
\]

(14)
for \( r = 1, 2, \ldots \) and \( k = 1, 2, \ldots, r \), where \( c_{2r,2k} = c_{2r-1,2k-1} = 0 \), for \( k = 1, 2, \ldots, r \), so that

\[
\sinh(2rx) = \sum_{k=1}^{r} c_{2r,2k+1} \cosh^{2r-2k+1} x \sinh x, \tag{15}
\]

\[
\sinh(2r-1)x = \sum_{k=1}^{r} c_{2r-1,2k} \cosh^{2r-2k} x \sinh x, \tag{16}
\]

for \( r = 1, 2, \ldots \) and \( k = 1, 2, \ldots, r \).

**Lemma 3.**

\[
\int \sinh^r x \, dx = \sum_{k=1}^{r} \sum_{i=1}^{k} \frac{a_{r,k}b_{k,i}}{k} \cosh^i x \sinh x, \tag{17}
\]

for \( r = 1, 2, \ldots \).

**Proof.** Using equations (7) and (14), we have

\[
\int \sinh^r x \, dx = \sum_{k=1}^{r} \int [a_{r,k} \cosh(kx) + b_{r,k} \sinh(kx)] \, dx
\]

\[= \sum_{k=1}^{r} \frac{a_{r,k} \sinh(kx) + b_{r,k} \cosh(kx)}{k}
\]

\[= \sum_{k=1}^{r} \sum_{i=1}^{k} \frac{a_{r,k}b_{k,i}}{k} \cosh^i x \sinh x,
\]

proving equation (17).

**Theorem 5.** The convolution \( \sinh^{-1} x_+ \ast x_+^r \) exists and

\[
\sinh^{-1} x_+ \ast x_+^r = \sum_{k=0}^{r} \binom{r}{k} \left[ (-1)^k x_+^{r+k+1} \sinh^{-1} x \right.
\]

\[\left. - \sum_{i=1}^{k+1} \sum_{j=1}^{i} \frac{a_{k+1,ij}b_{ji}}{i(k+1)} x_+^{r-k+1} (x^2 + 1)^{i/2} \right], \tag{18}
\]

for \( r = 0, 1, 2, \ldots \).

**Proof.** It is obvious that \( \sinh^{-1} x_+ \ast x_+^r = 0 \) if \( x < 0 \). When \( x > 0 \), we have

\[
\sinh^{-1} x_+ \ast x_+^r = \int_0^x \sinh^{-1} t(x - t)^r \, dt
\]

\[= \sum_{k=0}^{r} \binom{r}{k} x_+^{r-k} \int_0^x (-t)^k \sinh^{-1} t \, dt. \tag{19}
\]
Making the substitution \( t = \sinh u \), we get
\[
\int_0^x t^k \sinh^{-1} t \, dt = \int_0^{\sinh^{-1} x} u \sinh^k u \cosh u \, du
\]
\[
= \frac{x^{k+1} \sinh^{-1} x}{k+1} - \int_0^{\sinh^{-1} x} \frac{\sinh^{k+1} u}{k+1} \, du
\]
\[
= \frac{x^{k+1} \sinh^{-1} x}{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i}b_{i,j}}{i(k+1)} x(x^2+1)^{j/2},
\]
(20)
on using equation (17). Equation (18) now follows from equations (19) and (20).
□

Replacing \( x \) by \(-x\) in equation (18), we get

**Corollary 1.** The convolution \( \sinh^{-1} x_+ \ast x_+ \) exists and

\[
\sinh^{-1} x_+ \ast x_+ = \sum_{k=0}^\infty \binom{r}{k} \left[ \frac{(-1)^k x^{r+1} \sinh^{-1} x}{k+1} \right.
\]
\[
- \sum_{i=1}^{k+1} \sum_{j=1}^i \frac{a_{k+1,i}b_{i,j}}{i(k+1)} x^{-k+1}(x^2+1)^{j/2} \right],
\]
(21)
for \( r = 0, 1, 2, \ldots \)

The definition of the convolution is rather restrictive and so the non-commutative neutrix convolution was introduced in [2]. In order to define the neutrix convolution we first of all let \( \tau \) be a function in \( \mathcal{D} \) satisfying the following properties:

(i) \( \tau(x) = \tau(-x) \),
(ii) \( 0 \leq \tau(x) \leq 1 \),
(iii) \( \tau(x) = 1 \) for \( |x| \leq \frac{1}{2} \),
(iv) \( \tau(x) = 0 \) for \( |x| \geq 1 \).

The function \( \tau_n \) is then defined by
\[
\tau_n(x) = \begin{cases} 
1, & |x| \leq n, \\
\tau(n^nx - n^{n+1}), & x > n, \\
\tau(n^nx + n^{n+1}), & x < -n 
\end{cases}
\]
for \( n = 1, 2, \ldots \).

The following definition was given in [2].

**Definition 3.** Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) and let \( f_n = f \tau_n \) for \( n = 1, 2, \ldots \). Then the **neutrix convolution** \( f \circledast g \) is defined as the neutrix
limit of the sequence \( \{ f_n \ast g \} \), provided that the limit \( h \) exists in the sense

\[
N - \lim_{n \to \infty} \langle f_n \ast g, \varphi \rangle = \langle h, \varphi \rangle
\]

for all \( \varphi \) in \( \mathcal{D} \), where \( N \) is the neutrix, see van der Corput [1], having domain \( N' = \{ 1, 2, \ldots, n, \ldots \} \) and range \( N'' \), the real numbers, with negligible functions being finite linear sums of the functions

\[
n^\lambda \ln^{r-1} n, \ln^r n \quad (\lambda > 0, r = 1, 2, \ldots)
\]

and all functions which converge to zero in the usual sense as \( n \) tends to infinity.

In particular, if

\[
\lim_{n \to \infty} \langle f_n \ast g, \varphi \rangle = \langle h, \varphi \rangle
\]

for all \( \varphi \) in \( \mathcal{D} \), we say that the convolution \( f \ast g \) exists and equals \( h \).

Note that in this definition the convolution \( f_n \ast g \) is as defined in Gel’fand and Shilov’s sense, the distribution \( f_n \) having compact support. Note also that because of the lack of symmetry in the definition of \( f \otimes g \), the neutrix convolution is in general non-commutative.

The following theorem was proved in [2], showing that the neutrix convolution is a generalization of the convolution.

**Theorem 6.** Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) satisfying either condition (a) or condition (b) of Gel’fand and Shilov’s definition. Then the neutrix convolution \( f \otimes g \) exists and

\[
f \otimes g = f \ast g.
\]

We now prove the following theorem.

**Theorem 7.** The neutrix convolution \( \sinh^{-1} x_+ \otimes x^r \) exists and

\[
\sinh^{-1} x_+ \otimes x^r = \sum_{k=0}^{r} \binom{r}{k} (-1)^k x^{r-k} \left[ c_{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^{i} a_{k+1,i} b_{i,j} d_j \right],
\]

for \( r = 0, 1, 2, \ldots \), where

\[
c_k = \begin{cases} 0, & k \text{ odd}, \\ -\left( \frac{1}{k/2} \right)^{1/2}, & k \text{ even}, \end{cases}
\]

\[
d_k = \begin{cases} 0, & j \text{ even}, \\ \left( \frac{1}{(j+1)/2} \right), & j \text{ odd}. \end{cases}
\]
Proof. Putting \[ \sinh^{-1} x_+ \]_n = \sinh^{-1} x_+ \tau \_n(x), we have
\[
\left[ \sinh^{-1} x_+ \right]_n \star x^r = \int_0^n \sinh^{-1} t(x - t)^r dt + \int_{n}^{n+n-n} \sinh^{-1} t(x - t)^r \tau \_n(t) dt
\]
\[
= \sum_{k=0}^{r} \binom{r}{k} (-1)^k x^r - k \int_0^n t^k \sinh^{-1} t dt
\]
\[
+ \int_{n+n-n}^{n+n-n} \sinh^{-1} t(x - t)^r \tau \_n(t) dt
\]
\[
= I_1 + I_2. \tag{23}
\]
Replacing \( x \) by \( n \) in equation (20), we get
\[
\int_0^n t^k \sinh^{-1} t dt = \frac{n^{k+1} \sinh^{-1} n}{k + 1} - \sum_{i=1}^{k+1} \sum_{j=1}^{i} \frac{a_{k+1, i} b_i j}{i(k + 1)} n(n^2 + 1)j/2. \tag{24}
\]
Now,
\[
\left[ \sinh^{-1} x \right]' = (x^2 + 1)^{-1/2} = x^{-1} \sum_{i=0}^{\infty} \left( -\frac{1}{2} \right)_i x^{-2i}
\]
and so
\[
\sinh^{-1} x = \ln x - \sum_{i=1}^{\infty} \left( -\frac{1}{2} \right)_i \frac{x^{-2i}}{2i} + \text{const.} \tag{25}
\]
Hence, for \( k = 0, 1, 2, \ldots, \) we have
\[
N - \lim_{n \to \infty} n^k \sinh^{-1} n = \begin{cases} 0, & k \text{ odd,} \\ -\left( -\frac{1}{2} \right)_k \frac{1}{k}, & k \text{ even} \end{cases} = c_k, \tag{26}
\]
for short.

Further,
\[
(n^2 + 1)^j/2 = n^j \sum_{i=0}^{\infty} \left( \frac{j/2}{i} \right) n^{-2i}
\]
and so for \( j = 1, 2, \ldots, \) we have
\[
N - \lim_{n \to \infty} n(n^2 + 1)^j/2 = \begin{cases} 0, & j \text{ even,} \\ \left( \frac{j}{j+1/2} \right), & j \text{ odd} \end{cases} = d_j, \tag{27}
\]
for short.
It now follows from equations (24) to (26) that

\[ N^{-\lim_{n \to \infty}} I_1 = \sum_{k=0}^{r} \binom{r}{k} (-1)^k x^{r-k} \left[ c_{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^{i} \frac{a_{k+1,i} b_{i,j} d_j}{i(k+1)} \right]. \]  

(28)

Next, it is easily seen that \( I_2 = O(n^{-n}) \) and so

\[ \lim_{n \to \infty} I_2 = 0. \]  

(29)

Equation (22) now follows from equations (23), (28) and (29).

Replacing \( x \) by \(-x\) in equation (22), we get

**Corollary 2.** The neutrix convolution \( \sinh^{-1} x_- \ast x^r \) exists and

\[ \sinh^{-1} x_- \ast x^r = -\sum_{k=0}^{r} \binom{r}{k} x^{r-k} \left[ c_{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^{i} \frac{a_{k+1,i} b_{i,j} d_j}{i(k+1)} \right], \]  

(30)

for \( r = 0, 1, 2, \ldots \).

**Corollary 3.** The neutrix convolution \( \sinh^{-1} x \ast x^r \) exists and

\[ \sinh^{-1} x \ast x^r = \sum_{k=0}^{r} \binom{r}{k} \left[ (-1)^k - 1 \right] x^{r-k} \left[ c_{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^{i} \frac{a_{k+1,i} b_{i,j} d_j}{i(k+1)} \right], \]  

(31)

for \( r = 0, 1, 2, \ldots \).

*Proof.* We have

\[ \sinh^{-1} x \ast x^r = \sinh^{-1} x_+ \ast x^r + \sinh^{-1} x_- \ast x^r \]

and then equation (31) follows from equations (22) and (30).

**Corollary 4.** The neutrix convolution \( \sinh^{-1} x_+ \ast x_- \) exists and

\[ \sinh^{-1} x_+ \ast x_- = \sum_{k=0}^{r} \binom{r}{k} \left[ (-1)^k x^{r+k} \right] x^{-k} \left[ c_{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^{i} \frac{a_{k+1,i} b_{i,j} d_j}{i(k+1)} \right] \]

\[ - \sum_{k=0}^{r} \binom{r}{k} \left[ \frac{(-1)^k x^{r+k}}{k+1} \right] \sinh^{-1} x \]

\[ - \sum_{i=1}^{k+1} \sum_{j=1}^{i} \frac{a_{k+1,i} b_{i,j} d_j}{i(k+1)} x^{r-k+1} (x^2 + 1)^{i/2}, \]  

(32)

for \( r = 0, 1, 2, \ldots \).
Proof. We have

\[ (-1)^r \sinh^{-1} x_+ \odot x_+^r = \sinh^{-1} x_+ \odot x^r - \sinh^{-1} x_+ \ast x_+^r \]

\[ = \sum_{k=0}^{r} \binom{r}{k} (-1)^k x^{r-k} \left[ c_{k+1} - \sum_{i=1}^{k+1} \left( \sum_{j=1}^{i} a_{k+1,i} b_{i,j} d_{j} \right) \right] \]

\[ - \sum_{k=0}^{r} \binom{r}{k} \left[ (-1)^k x^{r+1} \sinh^{-1} x ight] \]

\[ - \sum_{i=1}^{k+1} \left( \sum_{j=1}^{i} a_{k+1,i} b_{i,j} x^{r-k+1} (x^2 + 1)^{j/2} \right) \]

on using equations (18 and (22) and equation (32) follows. □

Replacing \( x \) by \( -x \) in equation (32), we get

**Corollary 5.** The neutrix convolution \( \sinh^{-1} x_- \odot x_+^r \) exists and

\[ \sinh^{-1} x_- \odot x_+^r = \sum_{k=0}^{r} \binom{r}{k} x^{r-k} \left[ c_{k+1} - \sum_{i=1}^{k+1} \left( \sum_{j=1}^{i} a_{k+1,i} b_{i,j} d_{j} \right) \right] \]

\[ - \sum_{k=0}^{r} \binom{r}{k} \left[ (-1)^{r+k} x^{r+1} \sinh^{-1} x \right] \]

\[ + \sum_{i=1}^{k+1} \left( \sum_{j=1}^{i} a_{k+1,i} b_{i,j} \right) \left( -1 \right)^{r+j+k} x^{r-k+1} (x^2 + 1)^{j/2} \], (33)

for \( r = 0, 1, 2, \ldots \)

For further related results, see [4], [5], [7] and [8].

**References**


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