COMPLETE SEMIGROUPS OF BINARY RELATIONS DEFINED BY SEMILATTICES OF THE CLASS Z–ELEMENTARY X–SEMILATTICE OF UNIONS

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Abstract. In this paper we investigate idempotents of complete semigroups of binary relations defined by semilattices of the class Z–elementary X–semilattice of unions. For the case where X is a finite set we derive formulas by calculating the numbers of idempotents of the respective semigroup.

1. Introduction

Let X be an arbitrary nonempty set, D be an X–semilattice of unions, i.e. a nonempty set of subsets of the set X that is closed with respect to the set-theoretic operations of unification of elements from D, f be an arbitrary mapping from X into D. To each such a mapping f there corresponds a binary relation \( \alpha_f \) on the set X that satisfies the condition

\[ \alpha_f = \bigcup_{x \in X} (\{x\} \times f(x)) \]

The set of all such \( \alpha_f \) (\( f : X \to D \)) is denoted by \( B_X(D) \). It is easy to prove that \( B_X(D) \) is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by an X–semilattice of unions D.

We denote by \( \emptyset \) an empty binary relation or empty subset of the set X. The condition \((x,y) \in \alpha\) will be written in the form \( xy \). Further let \( x,y \in X, Y \subseteq X, \alpha \in B_X(D), T \in D, \emptyset \neq D' \subseteq D, \bar{D} = \cup D = \bigcup_{Y \in D} Y \) and \( t \in \bar{D} \). Then by symbols we denote the following sets:

\[ y\alpha = \{x \in X \mid yax\}, \quad Y\alpha = \bigcup_{y \in Y} y\alpha, \quad 2^X = \{Y \mid Y \subseteq X\}, \quad X^* = 2^X \setminus \{\emptyset\} \]

\[ V(D,\alpha) = \{Y \alpha \mid Y \in D\}, \quad D'_T = \{T' \in D' \mid T \subseteq T'\} \]

\[ \tilde{D}'_T = \{T' \in D' \mid T' \subseteq T\}, \quad D'_t = \{Z' \in D' \mid t \in Z'\}, \quad l(D', T) = \cup \big(D' \setminus D'_T\big). \]

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By symbol $\land (D, D')$ we mean an exact lower bound of the set $D'$ in the semilattice $D$.

**Definition 1.1.** Let $\alpha \in B_X(D)$. If $\alpha \circ \alpha = \alpha$, then $\alpha$ is called an idempotent element of the semigroup $B_X(D)$.

**Definition 1.2.** We say that a complete $X$-semilattice of unions $D$ is an $XI$-semilattice of unions if it satisfies the following two conditions:

a) $\land (D, D_t) \in D$ for any $t \in D$;

b) $Z = \bigcup_{t \in Z} \land (D, D_t)$ for any nonempty $Z$ element of $D$.

**Definition 1.3.** Let $D$ be an arbitrary complete $X$-semilattice of unions, $\alpha \in B_X(D)$ and $Y^{\alpha}_T = \{x \in X \mid x\alpha = T\}$. If

$$V[\alpha] = \begin{cases} V(X^*, \alpha), & \text{if } \emptyset \notin D, \\ V(X^*, \alpha), & \text{if } \emptyset \in V(X^*, \alpha), \\ V(X^*, \alpha) \cup \{\emptyset\}, & \text{if } \emptyset \notin V(X^*, \alpha) \text{ and } \emptyset \in D, \end{cases}$$

then it is obvious that any binary relation $\alpha$ of a semigroup $B_X(D)$ can always be written in the form $\alpha = \bigcup_{T \in V[\alpha]} (Y^{\alpha}_T \times T)$. In the sequel, such a representation of a binary relation $\alpha$ will be called quasinormal.

Note that for a quasinormal representation of a binary relation $\alpha$, not all sets $Y^{\alpha}_T (T \in V[\alpha])$ can be different from the empty set. But for this representation the following conditions are always fulfilled:

a) $Y^{\alpha}_{T} \cap Y^{\alpha}_{T'} = \emptyset$, for any $T, T' \in D$ and $T \neq T'$;

b) $X = \bigcup_{T \in V[\alpha]} Y^{\alpha}_T$.

**Lemma 1.4.** [2. Equality 6.9] Let $Y = \{y_1, y_2, \ldots, y_k\}$ and $D_j = \{T_1, T_2, \ldots, T_j\}$ be sets, where $k \geq 1$ and $j \geq 1$. Then the number $s(k, j)$ of all possible mappings of the set $Y$ on any such subset of the set $D_j'$ such that $T_j \in D_j'$ can be calculated by the formula $s(k, j) = j^k - (j - 1)^k$.

**Lemma 1.5.** Let $D_j = \{T_1, T_2, \ldots, T_j\}$, $X$ and $Y$ be three such sets, that $\emptyset \notin Y \subseteq X$. If $f$ is such mapping of the set $X$, in the set $D_j$, for which $f(y) = T_j$ for some $y \in Y$, then the number $s$ of all those mappings $f$ of the set $X$ in the set $D_j$ is equal to $s = j^{X \backslash Y} \cdot \left( j^{|Y|} - (j - 1)^{|Y|} \right)$.

**Proof.** Let $f_1$ be a mappings of the set $X \backslash Y$ in the set $D_j$, then the number of all such mappings is equal to $j^{X \backslash Y}$.

Now let $f_2$ be all mappings of the set $Y$ in the set $D_j$, for which $f(y) = T_j$ for some $y \in Y$, then by Lemma 1.4 the number of all such mappings is equal to $j^{|Y|} - (j - 1)^{|Y|}$.

We define the mapping $f$ of the set $X$ in the set $D_j$ by

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in X \backslash Y \\ f_2(x), & \text{if } x \in Y \end{cases}$$

It is clear that the mapping $f$ satisfies all the conditions of the given Lemma.
Thus the number \( s \) of all such maps is equal to all number of the pair \((f_1, f_2)\). The number all such pair is equal to \( s = j^{[\mathcal{Y}]} \cdot (j^{[\mathcal{Y}]} - (j - 1)^{[\mathcal{Y}]} ) \). \( \Box \)

The following Theorems are well known (see, \([1, 2, 3, 4, 5, 6]\)).

**Theorem 1.6.** [2, Theorem 2.1] A binary relation \( \alpha \in B_X(D) \) is a right unit of this semigroup iff \( \alpha \) is idempotent and \( D = V(D, \alpha) \).

**Theorem 1.7.** [2, Theorem 2.6] Let \( D \) be a complete \( X \)-semilattice of unions. The semigroup \( B_X(D) \) possesses a right unit iff \( D \) is an \( XI \)-semilattice of unions.

**Theorem 1.8.** [1, Theorem 6.2.3], [5, Theorem 6] Let \( D, \Sigma(D) \), \( E_X^{(\gamma)}(Q) \) and \( I_D \) denote respectively the complete \( X \)-semilattice of unions, the set of all \( XI \)-subsemilattices of the semilattice \( D \), the set of all right units of the semigroup \( B_X(D) \) and the set of all idempotents of the semigroup \( B_X(D) \). Then for the sets \( E_X^{(\gamma)}(Q) \) and \( I_D \) the following statements are true:

a) If \( \emptyset \notin D \) and \( \Sigma_\emptyset(D) = \{ D' \in \Sigma(D) \mid \emptyset \in D' \} \) then

1. \( E_X^{(\gamma)}(Q) \cap E_X^{(\gamma)}(Q') = \emptyset \) for any elements \( Q \) and \( Q' \) of the set \( \Sigma_\emptyset(D) \) that satisfy the condition \( Q \neq Q' \);
2. \( I_D = \bigcup_{Q \in \Sigma_\emptyset(D)} E_X^{(\gamma)}(Q) \);
3. The equality \( |I_D| = \sum_{Q \in \Sigma_\emptyset(D)} \left| E_X^{(\gamma)}(Q) \right| \) is fulfilled for the finite set \( X \).

b) If \( \emptyset \notin D \), then

1. \( E_X^{(\gamma)}(Q) \cap E_X^{(\gamma)}(Q') = \emptyset \) for any elements \( Q \) and \( Q' \) of the set \( \Sigma(D) \) that satisfy the condition \( Q \neq Q' \);
2. \( I_D = \bigcup_{Q \in \Sigma(D)} E_X^{(\gamma)}(Q) \);
3. The equality \( |I_D| = \sum_{Q \in \Sigma(D)} \left| E_X^{(\gamma)}(Q) \right| \) is fulfilled for the finite set \( X \).

**Theorem 1.9.** [3] Let \( D = \{ \hat{D}, Z_1, Z_2, \ldots, Z_{n-1} \} \) be some finite \( X \)-semilattice of unions and \( C(D) = \{ P_0, P_1, \ldots, P_{n-1} \} \) be the family of sets of pairwise non-intersecting subsets of the set \( X \). If \( \varphi \) is a mapping of the semilattice \( D \) on the family of sets \( C(D) \) which satisfies the condition \( \varphi(\hat{D}) = P_0 \) and \( \varphi(Z_i) = P_i \) for any \( i = 1, 2, \ldots, n-1 \) and \( \hat{D} = D \setminus \{ T \in D \mid Z \subseteq T \} \), then the following equalities are valid:

\[
\hat{D} = P_0 \cup P_1 \cup \cdots \cup P_{n-1}, \quad Z_i = P_0 \cup \bigcup_{T \in \hat{D}_i} \varphi(T). \tag{*}
\]

In the sequel these equalities will be called formal.

It is proved that if the elements of the semilattice \( D \) are represented in the form \((*)\), then among the parameters \( P_i \ (i = 1, 2, \ldots, n-1) \) there exists a parameter that cannot be empty sets for \( D \). Such sets \( P_i \ (0 < i \leq n - 1) \) are called basis sources, whereas sets \( P_i \ (0 \leq j \leq n - 1) \) which can be empty sets too are called completeness sources.

It is proved that under the mapping \( \varphi \) the number of covering elements of the pre-image of a basis source is always equal to one, while under the mapping \( \varphi \) the
number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one (see, [[3]]).

Lemma 1.10. Let $D = \{ \hat{D}, Z_1, Z_2, \ldots, Z_{n-1} \}$ and $C(D) = \{ P_0, P_1, \ldots, P_{n-1} \}$ be the finite semilattice of unions and the family of sets of pairwise nonintersecting subsets of the set $X$; $\varphi = \left( \begin{array}{c} \hat{D} \ Z_1 \ Z_2 \ \ldots \ \ Z_{n-1} \\ P_0 \ P_1 \ P_2 \ \ldots \ P_{n-1} \end{array} \right)$ is a mapping of the semilattice $D$ on the family of sets $C(D)$. If $\varphi(T) = P \in C(D)$ for some $\hat{D} \neq T \in D$, then $D_t = D \backslash \hat{D} T$ for all $t \in P$.

Proof. Let $t$ and $Z'$ be any elements of the set $P$ ($P \neq P_0$) and of the semilattice $D$ respectively. Then the equality $P \cap Z' = \emptyset$ (i.e., $Z' \notin D_t$ for any $t \in P$) is valid if and only if $T \notin \hat{D} Z'$ (if $T \in \hat{D} Z'$, then $\varphi(T) \subseteq Z'$ by definition of the formal equalities of the semilattice $D$). Since $\hat{D} Z' = D \backslash \{ T' \in D \mid Z' \subseteq T' \}$ by definition of the set $\hat{D} Z'$. Thus the condition $T \notin \hat{D} Z'$ hold iff $T \in \{ T' \in D \mid Z' \subseteq T' \}$. So, $Z' \subseteq T$ and $Z' \in \hat{D} T$ by definition of the set $\hat{D} T$.

Therefore, $\varphi(T) \cap Z' = \emptyset$ if and only if $Z' \in \hat{D} T$. Of this follows that the inclusion $\varphi(T) = P \subseteq Z'$ is true iff $D_t = D \backslash \hat{D} T$ for all $t \in \varphi(T) = P$. □

2. Results

Definition 2.1. Let $D$ be complete $X$–semilattice of unions and $Z$ be some fixed element of $D$. We say that a complete $X$–semilattice of unions $D$ is $Z$–elementary if $D$ satisfies the following conditions:

a) $D$ is not a chain;
b) every subchain of the semilattice $D$ is finite;
c) the set $D_Z = \{ T \in D \mid Z \subseteq T \}$ is a chain with smallest element $Z$;
d) the condition $T \cup T' = Z$ holds for any incomparable elements $T$ and $T'$ of $D$.

Example 1. The diagrams 1, 2, 3, 4 of the Fig. 2.1 respectively are $Z_1$, $\hat{D}$, $Z_1$ and $Z_2$ elementary $X$–semilattices of unions:

![Fig. 2.1](image)

Lemma 2.2. If $D$ is $Z$–elementary $X$–semilattice of unions, then $D \backslash \{ Z \}$ is unique generated set of the semilattice $D$.

Proof. The given Lemma immediately follows from the $Z$–elementary $X$–semilattice of unions. □
Lemma 2.3. Let $D$ be $Z$–elementary $X$–semilattice of unions. If subsemilattice $D'$ of the semilattice $D$ is not a chain, then $D'$ is $Z$–elementary $X$–semilattice of unions.

Proof. Let $D$ be $Z$–elementary $X$–semilattice of unions. Suppose that the subsemilattice $D'$ of the semilattice $D$ is not a chain.

1) It is clear, that the length of any chain of the semilattice $D'$ is finite since $D' \subseteq D$.

2) If $T \in D'_Z \setminus \{Z\}$, then $T \in D_Z$ since $T \in D' \subseteq D$, $Z \subseteq T$. We have $D'_Z \subseteq D_Z$.

3) Further, let $T$ and $T'$ be such elements of the set $D'$ such that $T \setminus T' \neq \emptyset$ and $T' \setminus T \neq \emptyset$ (i.e., the elements $T$ and $T'$ of $D$ are incomparable). Then $T,T' \in D$, since $D' \subseteq D$. From this we have $T \cup T' = Z$ by the definition of the $Z$–elementary $X$–semilattice union $D$.

From the conditions (1), (2) and (3) it follows, that $D'$ is $Z$–elementary $X$–semilattice of unions. □

Definition 2.4. Let $C$ and $C'$ be finite different chains of the set $2^X$ and $Z \in C \cap C'$. We say that the chains $C$ and $C'$ are $Z$–compatible if $C$ and $C'$ satisfy the following conditions:

a) $T \cup T' = Z$ for any $T \in C \setminus C'$ and $T' \in C' \setminus C$;

b) if $\overline{C}_Z = \{T \in C \mid Z \subseteq T\}$ and $\overline{C}'_Z = \{T' \in C' \mid Z \subseteq T'\}$, then $\overline{C}_Z = \overline{C}'_Z$ (see diagram 1 and 2 of the Fig.2.2).

Definition 2.5. The chain $C$ of a $X$–semilattice of unions $D$ is called maximal, if the inclusion $C \subseteq C'$ implies that $C = C'$ for any chain $C'$ of the $X$–semilattice of unions $D$.

Proof. Let $D$ be $Z$–elementary $X$–semilattice of unions and $C, C'$ be two different maximal subchains of the $X$–semilattice of unions $D$.

1) Let $\overline{C}_Z = \{T \in C \mid Z \subseteq T\}, \overline{C}'_Z = \{T' \in C' \mid Z \subseteq T'\}$. By assumption the sets $\overline{C}_Z, \overline{C}'_Z$ and $D_Z$ are maximal chains of the $X$–semilattice of union $D$ with smallest element $Z$. Then $D_Z = \overline{C}_Z = \overline{C}'_Z$ since by definition of the $Z$–elementary $X$–semilattice of unions $D$ the maximal subchains $D_Z, \overline{C}_Z, \overline{C}'_Z$ of the $X$–semilattice $D$, with the smallest element $Z$ are by definition unique.

2) Let $T \in C \setminus C'$ and $T' \in C' \setminus C$. Then $T \subset Z$, $T' \subset Z$ and $T \neq T'$. If $T \setminus T' \neq \emptyset$ and $T' \setminus T \neq \emptyset$ then $T \cup T' = Z$ by definition of the semilattice $D$. Therefore, the chains $C$ and $C'$ are $Z$–compatible.

1) So, we can assumed that $T \setminus T' = \emptyset$ and $T' \setminus T \neq \emptyset$. Further, let the element $T'$ cover the element $T'_1$ in the chain $C'$. Then $T' \setminus T' \neq \emptyset$ or $T' \setminus T \neq \emptyset$. If $T' \setminus T \neq \emptyset$, then we have $T' \setminus T \supset T' \setminus T' \neq \emptyset$. But the inequality $T' \setminus T \neq \emptyset$ contradicts the equality $T' \setminus T = \emptyset$. So, $T' \setminus T = \emptyset$.

2) Let $T'_1 \setminus T = \emptyset$. Then $T \supset T'_1$ and continue this process we obtain $T \supset T'_1 \supset T'_2 \supset \cdots \supset T'_q$ and $T'_q \setminus T = \emptyset$, where $T, T'_1, T'_2, \ldots, T'_q \in C'$ and element $T'_i$ cover the element $T'_{i+1}$ ($i = 1, 2, \ldots, q - 1$). But, this process must stop, since the chains $C'$ is finite. So, there exists a natural number $s$, such that $T \supset T'_1 \supset T'_2 \supset \cdots \supset T'_s$ and $T'_s \setminus T \neq \emptyset$. We have $T' \setminus T \supset T' \setminus T' \supset T' \setminus T' \supset \cdots \supset T' \setminus T$, i.e., $T' \setminus T \neq \emptyset$. We have $T' \setminus T \neq \emptyset, T' \setminus T' \neq \emptyset$ and $T' \setminus T = Z$ by definition of the $Z$–elementary $X$–semilattice of unions $D$. So, $T \cup T' = Z$ for any $T \in C \setminus C'$ and $T' \in C' \setminus C$.

Therefore, the chains are compatible.

Let any two maximal subchains of the $X$–semilattice of unions $D$ be $Z$–compatible. Then we have:

1) By supposition $D$ is not a chain.
2) Every subchain of the semilattice $D$ is finite, since all $Z$–compatible chains are finite.
3) By the definition of $Z$–compatible chains, the set $D_Z = \{T \in D \mid Z \subseteq T\}$ is a chain with smallest element $Z$. 
4) If $T$ and $T'$ are any incomparable elements of $D$, then there exist two maximal different chains $C$ and $C'$ such that $T \in C \setminus C'$, $T' \in C' \setminus C$ and the chains $C$ and $C'$ are $Z$–compatible, then $T \cup T' = Z$. \hfill $\Box$

Let $D$ be $Z$–elementary $X$–semilattice of unions and

$$Q = \{T_0, T_1, \ldots, T_{j-1}, T_j, T_{j+1}, \ldots, T_{k-1}, T_k, T_{k+1}, \ldots, T_{m-1}, T_m\}$$

be an $X$1–subsemilattice of the $Z$–elementary $X$–semilattice of unions $D$ which satisfy the following conditions:

$$T_0 \subseteq T_1 \subseteq \cdots \subseteq T_{j-1} \subseteq T_j \subseteq T_{j+1} \subseteq \cdots \subseteq T_{k-2} \subseteq T_k \subseteq T_{k+1} \subseteq \cdots \subseteq T_{m-1} \subseteq T_m,$$

$$T_0 \subseteq T_1 \subseteq \cdots \subseteq T_{j-1} \subseteq T_j \subseteq T_{j+2} \subseteq \cdots \subseteq T_{k-1} \subseteq T_k \subseteq T_{k+1} \subseteq \cdots \subseteq T_{m-1} \subseteq T_m,$$

$$T_{jk} \setminus T_{sj} \neq \emptyset, T_{sj} \setminus T_{jk} \neq \emptyset, T_{jk} \cup T_{sj} = T_{j+k+s}.$$
(i.e., the number different elements covered by the element $Z$ is two). Note that the diagram of the semilattice $D$ is shown in Fig. 2.3.

Further, let

$$C(Q) = \{ P_i \mid i = 0, 1, \ldots, j, j + k + s, j + k + s + 1, \ldots, m - 1, m \}$$

\[ \cup \{ P_{j1}, \ldots, P_{jk} \} \cup \{ P_{1j}, \ldots, P_{sj} \} \]

be a family of sets, where every two elements are pairwise disjoint subsets of the set $X$,

$$\varphi = \left( T_0 \ T_1 \ \cdots \ T_j \ T_{j1} \ \cdots \ T_{jk} \ T_{j1} \ \cdots \ T_{sj} \ T_{j+k+s} \ T_{j+k+s+1} \ \cdots \ T_{m-1} \ T_m \right).$$

Then for the formal equalities of the semilattice $Q$ we have:

$$T_m = P_m \cup P_{m-1} \cup \cdots \cup P_{j+k+s+1} \cup P_{j+k+s} \cup P_{sj} \cup \cdots \cup P_{j} \cup P_{jk} \cup \cdots \cup P_{j1} \cup P_{1j} \cup P_{1} \cup P_{0}$$

$$T_m = P_m \cup P_{m-2} \cup \cdots \cup P_{j+k+s+1} \cup P_{j+k+s} \cup P_{sj} \cup \cdots \cup P_{j} \cup P_{jk} \cup \cdots \cup P_{j1} \cup P_{1j} \cup P_{1} \cup P_{0}$$

Then the elements $P_0, P_1, \ldots, P_{j-1}, P_{j1}, \ldots, P_{sj}, P_{j+k+s}, P_{j+k+s+1}, \ldots, P_{m-1}$ are basis sources, the elements $P_j, P_m$ are sources of completeness of the semilattice $Q$.  

![Fig. 2.4](image1.png) ![Fig. 2.5](image2.png)
Lemma 2.7. Let $Q$ be a semilattice, whose diagram is shown in Fig. 2.3. Then $Q$ is a $XI$–semilattice of unions if and only if $k = s = 1$.

Proof. Let $t \in Q, Q_t = \{Z \in Q \mid t \in Z\}$ and $\bigwedge (Q, Q_t)$ be the exact lower bound of the set $Q_t$ in $Q$. Then from the formal equalities and by Lemma 1.10 we have:

$$
\begin{align*}
&\begin{array}{ll}
t \in P_m,& Q_t = Q, \\
t \in P_{m-1},& Q_t = Q \setminus \bar{Q}_{T_{m-1}} \\
\ldots&\ldots \\
t \in P_{j+k+s+1},& Q_t = Q \setminus \bar{Q}_{T_{j+k+s+1}} \\
t \in P_{j+k+s},& Q_t = Q \setminus \bar{Q}_{T_{j+k+s}} \\
t \in P_{j+k},& Q_t = Q \setminus \bar{Q}_{T_{j+k}} \\
t \in P_{j+k-1},& Q_t = Q \setminus \bar{Q}_{T_{j+k-1}} \\
\ldots&\ldots \\
t \in P_{j+1},& Q_t = Q \setminus \bar{Q}_{T_{j+1}} \\
t \in P_{j},& Q_t = Q \setminus \bar{Q}_{T_{j}} \\
\ldots&\ldots \\
t \in P_{0},& Q_t = Q \setminus \bar{Q}_{T_{0}} \\
\end{array}
\end{align*}
$$

Thus we have obtained that $\bigwedge (Q, Q_t) \in Q$ for all $t \in T_m$. Let

$$
Q^\wedge = \{\bigwedge (Q, Q_t) \mid t \in T_m\}
$$

$$
= \{T_0, T_1, T_2, \ldots, T_j, T_{j+1}, \ldots, T_{j+k+s+1}, \ldots, T_{m-1}, T_m\}
$$

and $Q'$ be the semilattice of unions generated by the set $Q^\wedge$.

If $k \geq 2$ or $s \geq 2$, i.e., $T_{j2} \in Q$ or $T_{j2} \in Q$ then $T_{j2} \notin Q'$ or $T_{j2} \notin Q'$. So, if $k \geq 2$ or $s \geq 2$, then $Q$ is not $XI$–semilattice of unions.

If $k = s = 1$, then $T_{j1} \cup T_{j1} = T_{j2+1} \in Q'$, i.e., $Q' = Q$.

Therefore, $Q$ is $XI$–semilattice of unions.

\[\Box\]

Theorem 2.8. Let $D$ be a $Z$–elementary $X$–semilattice of unions and $Q$ be any $XI$–subsemilattice of the $X$–semilattice of unions $D$. Then for the $XI$–semilattice $Q$ we have:

a) $Q$ is a finite chain (see. diagram 1 of Fig. 2.4);

b) $Q = \{T, T', Z\} \cup Q'$, where $T$ and $T'$ are elements of the semilattice $D$ such that $T \cap T' = \emptyset$ and $Q' = \{T_1, T_2, \ldots, T_s\} \subseteq D_Z \setminus \{Z\}$ (see. diagram 2 of Fig. 2.4);

c) $Q = Q' \cup \{T, T', T'', Z\} \cup Q''$, where $T, T', T'' \in D, T \subseteq T', T \subseteq T''$, $T', T''$ are incomparable elements of $D$; $Q' = \emptyset$ or $Q'' = \emptyset$, or $Q'$, $Q''$ are subchains of the semilattice $D$ satisfying the conditions $Q' \subseteq D_Z \setminus \{Z\}$ and $Q'' \subseteq D_T \setminus \{T\}$ (see. diagram 3 of Fig. 2.4).

(a') If $Z \notin Q$ then by the definition of $Z$–elementary $X$–semilattice of unions it follows that $Q$ is a finite $X$–chain.

Now, let $Z \in Q$ and $T$ be the unique element of the semilattice $Q$ which is covered by the element $Z$; If $T_1$ and $T_2$ are any incomparable elements of the semilattice $Q$ satisfying the conditions $T_1 \subseteq T$ and $T_2 \subset T$, then by the definition $Z$–elementary $X$–semilattice of unions it follows that $Z = T_1 \cup T_2 \subseteq T$. The inclusion $Z \subseteq T$ contradicts the condition $T \subset Z$. So, we have $T_1$ and $T_2$ are comparable elements of the semilattice $Q$, i.e., $T_1 \subset T_2$ or $T_2 \subset T_1$. Therefore $Q$ is a finite $X$–chain. The statement $(a')$ is proved.

(b') Let $T, T'$ and $T''$ be different elements of the semilattice $Q$ which are covered by the element $Z$ in the semilattice $Q$. Then

$$Z = T \cup T' = T \cup T'' = T' \cup T''$$

1) If $T \cap T' = \emptyset$, then $T = Z \setminus T'$, $T' = Z \setminus T$ and

$$T = Z \setminus T' = (T' \cup T'') \setminus T' \subseteq T''$$

It follows, that $Z = T \cup T' \subseteq T'' \cup T'' = T''$, i.e., $T'' = Z$ since $T'' \subseteq Z$. But the equality $T'' = Z$ contradict, that $T''$ is an element which is covered by the element $Z$ in the semilattice $Q$.

2) Now suppose that the intersection any two different elements which are covered by the element $Z$ in the semilattice $Q$ is not empty.

It is clear that $T \neq \emptyset$ and $T = \bigcup_{t \in T} \cap (Q, Q_t)$, since $Q$ is $XI$–semilattice of unions. From Lemma 2.3 it follows that $Q$ is $Z$–elementary $X$–semilattice of unions. By the definition of the $Z$–elementary $X$–semilattice of unions $D$ immediately follows that $D \setminus \{Z\}$ is unique generated set of the semilattice $D$. It follows that $T = \land (Q, Q_t)$ for some $t' \in T$. On the other hand, $t' \in T \subset Z = T' \cup T''$, i.e., $t' \in T'$ or $t' \in T''$. If $t' \in T'$, then we have $T' \subseteq Q_t$ and $T = \land (Q, Q_t) \subseteq T'$. The inclusion $T \subset T'$ contradicts the assumption that element $T$ is covered by the element $Z$ in the semilattice $Q$. This contradiction shows that number the elements which are covered by the element $Z$ of the $XI$–semilattice $Q$ are less or equal two.

For the elements $T$ and $T'$ of the semilattice $Q$ we consider two case.

3) If $T$ and $T'$ are minimal elements of the $X$–semilattice unions $Q$, $T \cap T' = \emptyset$ and $Q' = Q \setminus \{T, T', Z\}$, then $Q = \{T, T', Z\} \cup Q'$, where $Q' = \{T_1, T_2, \ldots, T_s\} \subseteq D_Z \setminus \{Z\}$ and $Q'$ is a chain by definition of $Z$–elementary $X$–semilattice of unions and $Q$. The statement $(b')$ is proved.

c') Now suppose that the elements $T'$ and $T''$ covered by the element $Z$ in the semilattice $Q$ are not minimal elements of the semilattice $Q$, i.e., $T \subseteq T'$ and $T \subset T''$ for some $T \in Q$. Then by Lemma 2.7 we have the element $T$ covered by the elements $T'$ and $T''$ in the semilattice $Q$. It is clear, that the set $\{T, T', T'', Z\}$ is a $X$–subsemilattice of the semilattice $Q$. 

Further, let $Q' = \{Z' \in Q \mid Z \subset Z'\}$ and $Q'' = Q\setminus \{Q' \cup \{T, T', T'', Z\}\}$. Then we have
\[ Q = Q' \cup \{T, T', T'', Z\} \cup Q''. \]
It is clear that $Q' \subseteq D_Z \setminus \{Z\}$ and is a subchain of the chain $D_Z$.

Now, let $Z''$ be any element of the set $Q''$. Then $Z'' \in Q$, $Z'' \notin Q' \cup \{T, T', T'', Z\}$ and $Z'' \subset T'$ or $Z'' \subset T''$ since $T'$ and $T''$ are maximal elements of the set $Q\setminus \{Q' \cup \{Z\}\}$. If $Z''$ and $T$ are incomparable elements of the semilattice $Q$ then $Z = T \cup Z'' \subseteq T'$ by the definition of $Z$–elementary $X$–semilattice unions and by the conditions $T \subset T'$ and $Z'' \subset T'$. But the inclusion $Z \subset T'$ contradicts the conditions $T' \subset Z$. So, $Z''$ and $T$ are comparable elements of the semilattice $Q$. From this follows that $Z'' \subset T$.

In the case $Z'' \subset T''$ we can similarly prove that $Z'' \subset T$.

Further let $Z''_1$ and $Z''_2$ are any incomparable elements of the set $Q''$ satisfying the conditions $Z''_1 \subset T$ and $Z''_2 \subset T$. Then by the definition $Z$–elementary $X$–semilattice of unions it follows that $Z = Z''_1 \cup Z''_2 \subseteq T$. The inclusion $Z \subset T$ contradicts the condition $T \subset Z$. So, we have $Z''_1$ and $Z''_2$ are comparable elements of the set $Q''$, i.e., $Z''_1 \subset Z''_2$ or $Z''_2 \subset Z''_1$. Therefore $Q''$ is a finite $X$–chain for which $\tilde{Q} \subseteq \tilde{D}_T \setminus \{T\}$. The statement (c) is proved.

**Definition 2.9.** Let $C(D)$ denote the set all chains of the $Z$–elementary $X$–semilattice unions $D$. $N(D) = \{|C| \mid C \in C(D)\}$ be the largest natural number of the set $N(D)$,
\[
C_k(D) = \{C \in C(D) \mid |C| = k\} \quad (1 \leq k \leq h(D)),
\]
\[
I_{C_k(D)} = \{\alpha \in B_X(D) \mid \alpha \circ \alpha = \alpha, \ V(D, \alpha) \subseteq C_k(D)\},
\]
\[
I_{C(D)} = \{\alpha \in B_X(D) \mid \alpha \circ \alpha = \alpha, \ V(D, \alpha) \subseteq C(D)\}.
\]

It is easy to see, that: $C(D) = C_1(D) \cup C_2(D) \cup \cdots \cup C_{h(D)}(D)$.

**Theorem 2.10.** Let $Q = \{T_0, T_1, \ldots, T_m\}$ be a subsemilattice of the semilattice $D$ such that $T_0 \subset T_1 \subset \cdots \subset T_m$ (see Fig. 2.5). Then a binary relation $\alpha$ of the semigroup $B_X(D)$ that has a quasinormal representation of the form $\alpha = \bigcup_{n=0}^{m} (Y_1^{\alpha} \times T_i)$ is a right unit of the semigroup $B_X(Q)$ iff $Q = V(D, \alpha)$ and $Y_1^{\alpha} \cup Y_2^{\alpha} \cup \cdots \cup Y_p^{\alpha} \supseteq T_p, Y_q^{\alpha} \cap T_q \neq \emptyset$ for any $p = 1, 2, \ldots, m - 1$ and $q = 1, 2, \ldots, m$.

**Proof.** Let $Q = \{T_0, T_1, \ldots, T_m\}$ be a subsemilattice of the semilattice $D$ such that $T_0 \subset T_1 \subset \cdots \subset T_m$. Then the given Theorem immediately follows from the Theorem 1.6 and Corollary 3 of [5]. (see, also, Corollary 13.1.2 of [1]).

**Theorem 2.11.** Let $Q = \{T_0, T_1, \ldots, T_m\}$ be a subsemilattice of the semilattice $D$ such that $T_0 \subset T_1 \subset \cdots \subset T_m$. If $E_X^0(Q)$ is the set of all right units of the semigroup $B_X(Q)$, then
\[
E_X^0(Q) = \left(2^{|T_1 \setminus T_0|} - 1\right) \left(3^{|T_2 \setminus T_1|} - 2^{|T_3 \setminus T_1|}\right) \cdots \left((m + 1)^{|T_m \setminus T_{m-1}|} - m^{|T_m \setminus T_{m-1}|}\right)
\]
(see, Theorem 6.5 of [2] or Corollary 13.1.5 of [1]).
Theorem 2.13. Let $Q = \{T_1, T_2, \ldots, T_m\}$ be a subsemilattice of the semilattice $D$ such that $T_1, T_2 \notin \{\emptyset\}$, $T_1 \cap T_2 = \emptyset$, $T_1 \cup T_2 = T_3$, $T_3 \subset T_4 \subset \cdots \subset T_m$. Then the semigroup $B_X(Q)$ has right unit if $T_1 \cap T_2 = \emptyset$ (see [6], Theorem 1).

Theorem 2.14. Let $Q = \{T_1, T_2, \ldots, T_m\}$ be a subsemilattice of the semilattice $D$ such that $T_1, T_2 \notin \{\emptyset\}$, $T_1 \cap T_2 = \emptyset$, $T_1 \cup T_2 = T_3$, $T_3 \subset T_4 \subset \cdots \subset T_m$ (see Fig. 2.6). Then a binary relation $\alpha$ of the semigroup $B_X(Q)$ that has a quasinormal representation of the form $\alpha = \bigcup_{i=0}^{m} (Y_i^o \times T_i)$ is a right unit of the semigroup $B_X(Q)$ iff $Q = V(D, \alpha)$ and $Y_i^o \supseteq T_1, Y_2^o \supseteq T_2, Y_3^o \supseteq T_3, \ldots, Y_{m-1}^o \supseteq T_{m-1}$ and $Y_q^o \cap T_q \neq \emptyset$ for any $k = 1, 2, \ldots, m-1$ and $q = 4, 5, \ldots, m-1$ (see Corollary 13.2.3 of [1]).

![Fig. 2.6](image)

![Fig. 2.7](image)

Theorem 2.15. Let $Q = \{T_1, T_2, \ldots, T_m\}$ be a subsemilattice of the semilattice $D$ such that $T_1, T_2 \notin \{\emptyset\}$, $T_1 \cap T_2 = \emptyset$, $T_1 \cup T_2 = T_3$, $T_3 \subset T_4 \subset \cdots \subset T_m$. If $E_X^{(r)}(Q)$ is the set of all right units of the semigroup $B_X(Q)$, then

$$E_X^{(r)}(Q) = \left(4^{|T_4 \setminus T_3|} - 3^{|T_3 \setminus T_2|}\right)\left(5^{|T_5 \setminus T_4|} - 4^{|T_4 \setminus T_3|}\right)\cdots \left(m^{|T_m \setminus T_{m-1}|} - (m-1)^{|T_{m-1} \setminus T_{m-2}|}\right)\ m^{|X \setminus T_m|}$$

(see Corollary 13.2.1 of [1]).
Definition 2.16. Let $v = \{(T, T', T'') \mid T, T', T'' \in D, T \subset T', T \subset T'', T'' \setminus T' \neq \emptyset, T'' \setminus T'' \neq \emptyset\} \neq \emptyset$, $Q(T, T', T'', Q', Q'') = Q' \cup \{T, T', T'', Z\} \cup Q''$ and $C''(D)$ be set of all $Q(T, T', T'', Q', Q'')$, where $(T, T', T'') \in v$, $Q' = \emptyset$ or $Q'' = \emptyset$, or $Q'$, $Q''$ are subchains of the semilattice $D$ satisfying the conditions $Q' \subseteq D \setminus \{Z\}$ and $Q \subseteq D \setminus \{T\}$.

Further, let $C_{sk}(T, T', T'', D) = \{Q(T, T', T'', Q', Q'') \mid |Q'| = s, |Q''| = k\}$ $(Q(T, T', T'') \in v)$, $I_{C_{sk}}^e(T, T', T'', D) = \{\alpha \in B_X(D) \mid \alpha \circ \alpha = \alpha, V(D, \alpha) \subseteq C_{sk}(T, T', T'', D)\}$, $I_{C''}^e(D) = \{\alpha \in B_X(D) \mid \alpha \circ \alpha = \alpha, V(D, \alpha) \subseteq C''(D)\}$, where $0 \leq s \leq 2^{D \setminus \{Z\}}$ and $0 \leq k \leq 2^{D \setminus \{Z\}}$.

It is easy to see, that $C''(D) = \bigcup_{(T, T', T'') \in v} C_{sk}(T, T', T'', D)$.

Theorem 2.17. Let $Q = \{T_0, T_1, \ldots, T_m\} \ (m \geq 3)$ be a semilattice and $j$ be a fixed natural number such that $0 \leq j \leq m - 3$ and

$T_0 \subset T_1 \subset \cdots \subset T_j \subset T_{j+1} \subset T_{j+3} \subset \cdots \subset T_m,$

$T_0 \subset T_1 \subset \cdots \subset T_j \subset T_{j+2} \subset T_{j+3} \subset \cdots \subset T_m,$

$T_{j+1} \setminus T_{j+2} \neq \emptyset, \ T_{j+2} \setminus T_{j+1} \neq \emptyset, \ T_{j+1} \cup T_{j+2} = T_{j+3}$

(see Fig. 2.7). A binary representation $\alpha$ of the semigroup $B_X(Q)$, which has a quasinormal representation of the form $\alpha = \bigcup_{i=0}^m (Y_i \times T_i)$ such that $Q = V(D, \alpha)$, is an idempotent element of the semigroup $B_X(D)$ iff

$Y_0^\alpha \cup Y_1^\alpha \cup \cdots \cup Y_j^\alpha \supseteq T_{j+1} \cap T_{j+2},$

$Y_0^\alpha \cup Y_1^\alpha \cup \cdots \cup Y_j^\alpha \cup Y_{j+2}^\alpha \supseteq T_{j+2},$

$Y_0^\alpha \cup Y_1^\alpha \cup \cdots \cup Y_p^\alpha \supseteq T_p, \ Y_p^\alpha \cap T_p \neq \emptyset$

for any $p = 0, 1, 2, \ldots, m - 1, \ q = 1, 2, \ldots, m \ (p \neq j + 2, \ q \neq j + 3)$ (see Corollary 13.3.1 of [1]).

Theorem 2.18. Let $Q = \{T_0, T_1, \ldots, T_j, \ldots, T_m\} \ (m \geq 3)$ be a semilattice and $j$ be a fixed natural number such that $0 \leq j \leq m - 3$ and

$T_0 \subset T_1 \subset \cdots \subset T_j \subset T_{j+1} \subset T_{j+3} \subset \cdots \subset T_m,$

$T_0 \subset T_1 \subset \cdots \subset T_j \subset T_{j+2} \subset T_{j+3} \subset \cdots \subset T_m,$

$T_{j+1} \setminus T_{j+2} \neq \emptyset, \ T_{j+2} \setminus T_{j+1} \neq \emptyset, \ T_{j+1} \cup T_{j+2} = T_{j+3}$.

If $E_X^j(Q)$ is the set of all right units of the semigroup $B_X(Q)$, then the following statements are true:

a) $|E_X^j(Q)| = \left(2^{|T_1 \setminus T_2|} - 1\right) \left(2^{|T_2 \setminus T_3|} - 1\right) \left(5^{|T_k \setminus T_{k+1}|} - 4^{|T_k \setminus T_{k+1}|}\right) \cdots \left((m + 1)^{|T_m \setminus T_{m-1}|} - m^{|T_m \setminus T_{m-1}|}\right) (m + 1)^{|X \setminus T_m|},$

If $j = 0$ (i.e., $T_j = T_0$);
Let
\[
\text{Theorem 2.20.}
\]
If \( \alpha \) is an idempotent relation of the semigroup \( B_X(D) \) iff binary relation \( \alpha \) satisfies only one condition of the following conditions:

a) \( \alpha = (X \times T) \), where \( T \in D \):
b) \( \alpha = (Y_0^\alpha \times T_0) \cup (Y_1^\alpha \times T_1) \cup \cdots \cup (Y_k^\alpha \times T_k) \), where \( T_0, T_1, \ldots, T_k \in D \), \( T_0 \subseteq T_1 \subseteq \cdots \subseteq T_k \), \( 2 \leq k \leq b(D) \), \( Y_1^\alpha, \ldots, Y_k^\alpha \subseteq \{0\} \) and satisfies the conditions: \( Y_p^\alpha \cup Y_q^\alpha \cup \cdots \cup Y_p^\alpha \supseteq T_p \), \( Y_p^\alpha \cap T_q \neq \emptyset \) for any \( p = 0, 1, \ldots, k - 1 \) and \( q = 1, 2, \ldots, k \);

c) \( \alpha = (Y_0^\alpha \times T) \cup (Y_1^\alpha \times T) \cup (Y_2^\alpha \times Z) \), where \( T, T' \in D \), \( T_1 \cap T_2 = \emptyset \), \( Y_1^\alpha, Y_2^\alpha \subseteq \{0\} \) and satisfies the conditions: \( Y_T^\alpha \supseteq T \), \( Y_T'^\alpha \supseteq T' \);

d) \( \alpha = (Y_1^\alpha \times T_1) \cup (Y_2^\alpha \times T_2) \cup \cdots \cup (Y_s^\alpha \times T_s) \), where \( T_1, T_2, \ldots, T_s \in D \), \( T_1 = T, T_2 = T', T_3 = Z \), \( 4 \leq s \leq 2b(D) \), \( T_1 \cap T_2 = \emptyset \), \( Y_1^\alpha, Y_2^\alpha, \ldots, Y_s^\alpha \subseteq \{0\} \) and satisfies the conditions: \( Y_1^\alpha \supseteq T_1 \), \( Y_2^\alpha \supseteq T_2 \), \( Y_1^\alpha, Y_2^\alpha, \ldots, Y_p^\alpha \subseteq T_p \) and \( Y_q^\alpha \cap T_q \neq \emptyset \) for any \( p = 4, 5, \ldots, s - 1 \) and \( q = 4, 5, \ldots, s \);

e) \( \alpha = (Y_0^\alpha \times T_0) \cup (Y_1^\alpha \times T_1) \cup \cdots \cup (Y_j^\alpha \times T_j) \cup (Y_j^\alpha \times T_j + 1) \cup (Y_{j+1}^\alpha \times T_{j+2}) \cup (Y_{j+1}^\alpha \times T_{j+3}) \cup \cdots \cup (Y_{m-1}^\alpha \times T_m) \cup (Y_m^\alpha \times T_m) \), where \( T_0, T_1, T_2, T', T'', Z, T_{j+3}, \ldots, T_{m-1}, T_m \in D \), \( T_j = T, T_{j+1} = T', T_{j+2} = T'', T_{j+3} = Z, Y_0^\alpha, Y_1^\alpha, \ldots, Y_j^\alpha, Y_j^\alpha, Y_{j+1}^\alpha, Y_{j+2}^\alpha, Y_{j+3}^\alpha, \ldots, Y_m^\alpha \subseteq \{0\} \) and satisfies the conditions:

\[
Y_0^\alpha \cup Y_1^\alpha \cup \cdots \cup Y_j^\alpha \supseteq T_{j+2},
\]
\[
Y_0^\alpha \cup Y_1^\alpha \cup \cdots \cup Y_j^\alpha \cup Y_{j+2}^\alpha \supseteq T_{j+2},
\]
\[
Y_0^\alpha \cup Y_1^\alpha \cup \cdots \cup Y_p^\alpha \supseteq T_p, Y_q^\alpha \cap T_q \neq \emptyset
\]

Proof. The given Theorem immediately follows from Theorem 1.8. □

**Theorem 2.19.** If \( D \) is \( Z \)-elementary \( X \)-semilattice of unions, then the following equalities are true:

\[
|I_{C(D)}| = |I_{C_1(D)}| + |I_{C_2(D)}| + \cdots + |I_{C_k(D)}|,
\]
\[
|I_{C'}(D)| = |I_{C'_1(D)}| + |I_{C'_2(D)}| + \cdots + |I_{C'_{2x(D)}(Z)}(D)|,
\]
\[
|I_{C''(D)}| = \sum_{(T, T', T'') \in \mathcal{C}} |I_{C_{3k}(T, T', T'')}|.
\]
for any $p = 0, 1, \ldots, m - 1$, $q = 1, 2, \ldots, m$ ($p \neq j + 2$, $q \neq j + 3$) (see Corollary 13.3.1 of [1]).

Proof. The given Theorem immediately follows from the Theorem 2.10, 2.14 and 2.17. □

Theorem 2.21. Let $D$ and $I_D$ be any $Z$–elementary $X$–semilattice of unions and all idempotent elements of the $Z$–elementary $X$–semilattice of unions respectively. Then the following conditions are true.

a) $|I_D| = |I_{C(D)}|$, if $\mu = \emptyset$ and $\nu = \emptyset$;

b) $|I_D| = |I_{C(D)}| + |I_{C'(D)}|$ if $\mu \neq \emptyset$ and $\nu = \emptyset$;

c) $|I_D| = |I_{C(D)}| + |I_{C'(D)}|$ if $\mu = \emptyset$ and $\nu \neq \emptyset$;

d) $|I_D| = |I_{C(D)}| + |I_{C'(D)}| + |I_{C''(D)}|$ if $\mu \neq \emptyset$ and $\nu \neq \emptyset$.

Proof. The given Theorem immediately follows from the Theorem 2.19. □

Theorem 2.22. If $D$ is any $Z$–elementary $X$–semilattice of unions, then for any idempotent binary relation $\varepsilon$ from the semigroup $B_X(D)$ the order of maximal subgroup $G_X(D, \varepsilon)$ is not greater than two.

Proof. Let $D$ be any $Z$–elementary $X$–semilattice of unions and $\varepsilon \circ \varepsilon = \varepsilon$. As is known (see [1]) the group $G_X(D, \varepsilon)$ is anti-isomorphic to the group of all complete automorphisms of the semilattice $V(D, \varepsilon)$. In this case the number of all complete automorphisms of the semilattice $V(D, \varepsilon)$ is not greater than two. Therefore the order of maximal subgroup $G_X(D, \varepsilon)$ is not greater than two. □

Example 2. Let $D = \left\{ Z_4, Z_3, Z_2, Z_1, \bar{D} \right\}$ be $Z_1$–elementary $X$–semilattice of unions satisfying the conditions

$Z_3 \subset Z_2 \subset Z_1 \subset \bar{D}$, $Z_4 \subset Z_1 \subset D$, $Z_4 \setminus Z_3 \neq \emptyset$

$Z_4 \setminus Z_4 \neq \emptyset$, $Z_4 \setminus Z_2 \neq \emptyset$, $Z_4 \setminus Z_3 \neq \emptyset$

$Z_4 \cup Z_3 = Z_1$, $Z_3 \cup Z_2 = Z_1$. (2.1)

The semilattice satisfying the conditions (2.1) is shown in Fig. 2.8.

Let $C(D) = \left\{ P_0, P_1, P_2, P_3, P_4 \right\}$ be a family sets, where $P_0, P_1, P_2, P_3, P_4$ are pairwise disjoint subsets of the set $X$ and

$\varphi = \left( \begin{array}{cc}
\bar{D} & Z_1 \ Z_2 \ Z_3 \ Z_4 \\
P_0 & P_1 & P_2 & P_3 & P_4
\end{array} \right)$
is a mapping of the semilattice $D$ onto the family sets $C(D)$. Then for the formal equalities of the semilattice $D$ we have a form:

$$
\begin{align*}
&D = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \\
&Z_1 = P_0 \cup P_2 \cup P_3 \cup P_4 \\
&Z_2 = P_0 \cup P_3 \cup P_4 \\
&Z_3 = P_0 \cup P_2 \cup P_4 \\
&Z_4 = P_0 \cup P_3.
\end{align*}
$$

Here the elements $P_1, P_2, P_3, P_4$ are basic sources; the elements $P_0$ are sources of completeness of the $Z_1$–elementary $X$–semilattice of unions $D$.

Further, we have $Z_4 \cap Z_3 = (P_0 \cup P_3) \cap (P_0 \cup P_2 \cup P_4) = P_0$.

(1) If $Z_4 \cap Z_3 \neq \emptyset (P_0 \neq \emptyset)$, then $h(D) = 4, \mu = \nu = 0$

$$
\begin{align*}
C_1(D) &= \left\{ \{Z_4 \}, \{Z_3 \}, \{Z_2 \}, \{Z_1 \}, \{\bar{D} \} \right\} \\
C_2(D) &= \left\{ \{Z_4, Z_2 \}, \{Z_4, Z_3 \}, \{Z_4, \bar{D} \}, \{Z_3, Z_1 \}, \{Z_3, \bar{D} \}, \{Z_2, Z_1 \} \right\} \\
C_3(D) &= \left\{ \{Z_4, Z_2, \bar{D} \}, \{Z_4, Z_1, \bar{D} \}, \{Z_4, Z_3, \bar{D} \}, \{Z_4, Z_2, Z_1 \}, \{Z_3, Z_1, \bar{D} \} \right\} \\
C_4(D) &= \left\{ \{Z_4, Z_2, Z_1, \bar{D} \} \right\}
\end{align*}
$$

and $|I_{C_1(D)}| = |I_{C_2(D)}| + |I_{C_3(D)}| + |I_{C_4(D)}|$; where

$$
\begin{align*}
|I_{C_1(D)}| &= 5; \\
|I_{C_2(D)}| &= \left(2^{3|Z_4|} - 1\right) 2^{3|Z_2|} + \left(2^{3|Z_3|} + 2^{3|Z_1|} + 2^{3|Z_2| - 3}\right) 2^{3|X|} \\
+ \left(2^{3|Z_4|} + 2^{3|Z_3|} + 2^{3|Z_1|} - 4\right) 2^{3|X|}, \\
|I_{C_3(D)}| &= \left(2^{3|Z_4|} - 1\right) 2^{3|Z_2|} + 2^{3|Z_3|} - 2^{3|Z_1|} + 3^{3|X|} \\
+ \left(2^{3|Z_4|} - 1\right) \left(3^{3|Z_2|} - 2^{3|Z_1|}\right) 3^{3|X|}, \\
|I_{C_4(D)}| &= \left(2^{3|Z_4|} - 1\right) \left(3^{3|Z_2|} - 2^{3|Z_1|}\right) 4^{3|X|}.
\end{align*}
$$

(see Theorem 2.4).

If $X = \{1, 2, 3, 4, 5\}, D = \{\{3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}$
then $|I_{C_1(D)}| = 5, |I_{C_2(D)}| = 28, |I_{C_3(D)}| = 13, |I_{C_4(D)}| = 1, |IC(D)| = 47$.

(2) If $Z_4 \cap Z_3 = \emptyset (P_0 = \emptyset)$, then $\mu = \{\{Z_4, Z_3\}\}, \nu = 0, h(D) = 4, s = 0, 1$ and

$$
\begin{align*}
C_1(D) &= \left\{ \{Z_4 \}, \{Z_3 \}, \{Z_2 \}, \{Z_1 \}, \{\bar{D} \} \right\} \\
C_2(D) &= \left\{ \{Z_4, Z_2 \}, \{Z_4, Z_1 \}, \{Z_4, \bar{D} \}, \{Z_3, Z_1 \}, \{Z_3, \bar{D} \}, \{Z_2, Z_1 \} \right\}
\end{align*}
$$
Let $C(D) = \{ \{ Z_2, Z_1, \tilde{D} \} \cup \{ Z_4, Z_1, \tilde{D} \} \cup \{ Z_4, Z_2, \tilde{D} \} \cup \{ Z_3, Z_1, \tilde{D} \} \}$

$C(D) = \{ \{ Z_2, Z_1, \tilde{D} \} \cup \{ Z_4, Z_1, \tilde{D} \} \cup \{ Z_4, Z_2, \tilde{D} \} \cup \{ Z_3, Z_1, \tilde{D} \} \}.$

$C(D) = C_1(D) \cup C_2(D) \cup C_3(D) \cup C_4(D)$

$C_1(D) = \{ \{ Z_4, Z_2, Z_1 \} \}$

$C_2(D) = \{ \{ Z_3, Z_1, \tilde{D} \} \}$

$C_3(D) = \{ \{ Z_4, Z_1, \tilde{D} \} \}$

$\tilde{D}$ and $I_D = |I_{C_1(D)}| + |I_{C_2(D)}| + |I_{C_3(D)}| + |I_{C_4(D)}| + |I_{C_0(D)}| + |I_{C_1(D)}|,$ where

$I_{C_0(D)} = 5,
I_{C_1(D)} = (2|Z_2 \setminus Z_1| - 1) 2|X \setminus Z_2| + (2|Z_1 \setminus Z_2| + 2|Z_1 \setminus Z_3| + 2|Z_1 \setminus Z_2| - 3) 2|X \setminus Z_1|
+ (2|\tilde{D} \setminus Z_1| + 2|\tilde{D} \setminus Z_3| + 2|\tilde{D} \setminus Z_2| - 4) 2|X \setminus D|;
I_{C_2(D)} = (2|Z_2 \setminus Z_1| - 1) (3|Z_1 \setminus Z_2| - 3|D \setminus Z_1| + (2|Z_1 \setminus Z_2| - 1) + (2|Z_1 \setminus Z_2| - 1)
(3|Z_1 \setminus Z_2| - 2|D \setminus Z_2|) 3|X \setminus D|
+ (2|Z_2 \setminus Z_1| - 1) (3|Z_1 \setminus Z_2| - 2|Z_1 \setminus Z_3|) 3|X \setminus Z_1| + (2|Z_4 \setminus Z_2| - 1)
(3|Z_1 \setminus Z_3| - 2|D \setminus Z_3|) 3|X \setminus D|
+ |D \setminus Z_1| - 2|Z_1 \setminus Z_2|)
(I_{C_3(D)} = (2|Z_2 \setminus Z_1| - 1) (3|Z_1 \setminus Z_2| - 2|Z_1 \setminus Z_3|)
(3|Z_1 \setminus Z_2| - 2|D \setminus Z_1|) 4|X \setminus D|
I_{C_4(D)} = 3|X \setminus D|,
I_{C_0(D)} = 3|X \setminus Z_1|,
I_{C_1(D)} = 4|D \setminus Z_1| - 3|D \setminus Z_2|) 4|X \setminus D|
(see Theorems 2.11 and 2.15).

If $X = \{1, 2, 3, 4\}, D = \{\{3\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$ then $|I_{C_1(D)}| = 5,$
$I_{C_2(D)} = 28, I_{C_3(D)} = 13, I_{C_4(D)} = 1, I_{C_0(D)} = 3, I_{C_1(D)} = 1, |I_D| = 51.$

Example 3. Let $D = \{ Z_3, Z_4, Z_3, Z_2, Z_1, \tilde{D} \}$ be $Z_1$-elementary $X$-semilattice of unions satisfying the conditions

$Z_3 \subset Z_2 \subset Z_1 \subset D, Z_3 \subset Z_4 \subset Z_1 \subset \tilde{D}.$ $Z_3 \subset Z_4 \subset Z_1 \subset D, Z_3 \subset Z_4 \subset Z_1 \subset \tilde{D} \setminus Z_3 \neq \emptyset \setminus Z_1 \setminus Z_2 \neq \emptyset, Z_2 \setminus Z_1 \neq \emptyset, Z_3 \setminus Z_4 \neq \emptyset, Z_2 \setminus Z_3 \neq \emptyset.$

The semilattice satisfying the conditions (2.2) is shown in Fig. 2.9.

Let $C(D) = \{ P_0, P_1, P_2, P_3, P_4, P_5 \}$ be a family sets, where $P_0, P_1, P_2, P_3, P_4, P_5$
are pairwise disjoint subsets of the set $X$ and

$\varphi = \begin{pmatrix}
\tilde{D} & Z_1 & Z_2 & Z_3 & Z_4 & Z_5 \\
\hat{P_0} & \hat{P_1} & \hat{P_2} & \hat{P_3} & \hat{P_4} & \hat{P_5}
\end{pmatrix}$

be a mapping of the semilattice $D$ onto the family sets $C(D).$ Then for the formal equalities of the semilattice $D$ we have a form:

$\tilde{D} = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5$
$Z_1 = P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_5$
$Z_2 = P_0 \cup P_3 \cup P_4 \cup P_5$
Here the elements $P_1, P_2, P_3, P_4$ are basic sources; the elements $P_0, P_5$ are sources of completeness of the $Z_1$–elementary $X$–semilattice of unions $D$.

Further, we have $Z_0 \cap Z_3 = (P_0 \cup P_1) \cap (P_0 \cup P_2 \cup P_4 \cup P_5) = P_0$.

(1) If $Z_0 \cap Z_3 \neq \emptyset$ ($P_0 \neq \emptyset$), then $\mu = \emptyset$, $\nu = \{(Z_3, Z_4, Z_2)\}$, $h(D) = 4$, $s = 0, 1$,

$C_1(D) = \{\{Z_3\}, \{Z_4\}, \{Z_2\}, \{Z_3\}, \{Z_4\}\}$,

$C_2(D) = \{\{Z_3, Z_4\}, \{Z_3, Z_2\}, \{Z_3, Z_1\}, \{Z_4, Z_1\}\}$,

$C_3(D) = \{\{Z_3, Z_1\}, \{Z_3, Z_4\}, \{Z_3, Z_2\}, \{Z_3, Z_1\}\}$,

$C_4(D) = \{\{Z_3, Z_4, Z_1, D\}\}$,

$C_5'(D) = \{\{Z_3, Z_4, Z_2, Z_1, D\}\}$,

$C(D) = C_1(D) \cup C_2(D) \cup C_3(D) \cup C_4(D), C''(D) = C_5'(D) \cup C''(D)$.

and $|I_D| = |I_{C_1(D)}| + |I_{C_2(D)}| + |I_{C_3(D)}| + |I_{C_4(D)}| + |I_{C_5'(D)}| + |I_{C''(D)}|$, where

$|I_{C_1(D)}| = 6$;

$|I_{C_2(D)}| = 2^{(|Z_4| + |Z_3| + |Z_2| - 4)} + 2^{(|Z_4| + |Z_2| - 4)} + 2^{(|Z_3| + |Z_2| - 4)} + 2^{(|Z_4| + |Z_3| - 4)} + 2^{(|Z_3| + |Z_2| - 4)} + 2^{(|Z_4| + |Z_3| - 4)} + 2^{(|Z_3| + |Z_2| - 4)}$;

$|I_{C_3(D)}| = 2^{(|Z_4| + |Z_3| - 4)} + 2^{(|Z_4| + |Z_2| - 4)} + 2^{(|Z_3| + |Z_2| - 4)} + 2^{(|Z_4| + |Z_3| - 4)} + 2^{(|Z_3| + |Z_2| - 4)} + 2^{(|Z_4| + |Z_3| - 4)} + 2^{(|Z_3| + |Z_2| - 4)}$;

$|I_{C_4(D)}| = 2^{(|Z_4| + |Z_3| - 4)} + 2^{(|Z_4| + |Z_2| - 4)} + 2^{(|Z_3| + |Z_2| - 4)} + 2^{(|Z_4| + |Z_3| - 4)} + 2^{(|Z_3| + |Z_2| - 4)} + 2^{(|Z_4| + |Z_3| - 4)} + 2^{(|Z_3| + |Z_2| - 4)}$;

$|I_{C_5'(D)}| = 2^{(|Z_4| + |Z_3| - 4)} + 2^{(|Z_4| + |Z_2| - 4)} + 2^{(|Z_3| + |Z_2| - 4)} + 2^{(|Z_4| + |Z_3| - 4)} + 2^{(|Z_3| + |Z_2| - 4)} + 2^{(|Z_4| + |Z_3| - 4)} + 2^{(|Z_3| + |Z_2| - 4)}$;

$|I_{C''(D)}| = 2^{(|Z_4| + |Z_3| - 4)} + 2^{(|Z_4| + |Z_2| - 4)} + 2^{(|Z_3| + |Z_2| - 4)} + 2^{(|Z_4| + |Z_3| - 4)} + 2^{(|Z_3| + |Z_2| - 4)} + 2^{(|Z_4| + |Z_3| - 4)} + 2^{(|Z_3| + |Z_2| - 4)}$;

$(see Theorems 2.11 and 2.18).

If $X = \{1, 2, 3, 4, 5, 6\}$,

$D = \{(3, 6), (2, 3, 5, 6), (2, 4, 5, 6), (3, 4, 5, 6), (2, 3, 4, 5, 6), (1, 2, 3, 4, 5, 6)\}$

then $|I_{C_1(D)}| = 6$, $|I_{C_2(D)}| = 69$, $|I_{C_3(D)}| = 58$, $|I_{C_4(D)}| = 6$, $|I_{C_5'(D)}| = 4$, $|I_{C''(D)}| = 1$, $|I_D| = 144$. 

(2) If \(Z_5 \cap Z_1 = \emptyset\) \((P_0 = \emptyset)\), then \(\mu = \{(Z_5, Z_1)\}, \nu = \{(Z_5, Z_2, Z_4)\}, h(D) = 4, s = 0, 1,\)

\[C_1(D) = \left\{ \{Z_1\}, \{Z_4\}, \{Z_4\}, \{Z_4\}, \{Z_3\}, \{Z_1\}, \{\tilde{D}\} \right\},\]

\[C_2(D) = \left\{ \{Z_5, Z_2\}, \{Z_5, Z_1\}, \{Z_5, Z_1\}, \{Z_4, \tilde{D}\}, \{Z_4, Z_1\}, \{Z_3, Z_1\}, \{Z_3, \tilde{D}\}, \{Z_1, \tilde{D}\} \right\},\]

\[C_3(D) = \left\{ \{Z_5, Z_4, Z_1\}, \{Z_5, Z_4, \tilde{D}\}, \{Z_3, Z_2, Z_1\}, \{Z_5, Z_2, \tilde{D}\}, \{Z_5, Z_1, \tilde{D}\} \right\},\]

\[C_4(D) = \left\{ \{Z_5, Z_3, Z_1, \tilde{D}\}, \{Z_5, Z_3, \tilde{D}\} \right\},\]

\[C_{10}^0(D) = \left\{ \{Z_5, Z_3, Z_1\} \right\},\]

\[C_{10}^0(D) = \left\{ \{Z_5, Z_4, Z_3, Z_1, \tilde{D}\} \right\},\]

\[C(D) = C_1(D) \cup C_2(D) \cup C_3(D) \cup C_4(D),\]

\[C'(D) = C_1(D) \cup C_2(D) \cup C_3(D) \cup C_4(D),\]

\[C''(D) = C_1(D) \cup C_2(D) \cup C_3(D) \cup C_4(D),\]

and \(|I_D| = |I_{C_1(D)}| + |I_{C_2(D)}| + |I_{C_3(D)}| + |I_{C_4(D)}| + |I_{C_{10}^0(D)}| + |I_{C'(D)}| + |I_{C''(D)}| + |I_{C''(D)}|\), where

\[|I_{C_1(D)}^*| = 6;\]

\[I_{C_2(D)}^* = \left\{ \begin{array}{c}
2|\tilde{z}_1\tilde{z}_2| + 2|\tilde{z}_1\tilde{z}_3| + 2|\tilde{z}_1\tilde{z}_4| + 2|\tilde{z}_1\tilde{z}_5| - 4 \\
+ 2^{|\tilde{x}\tilde{z}_1|} - 1 \\
+ 2|\tilde{z}_2\tilde{z}_3| - 2|\tilde{z}_2\tilde{z}_4| - 1 \\
+ 2|\tilde{z}_2\tilde{z}_5| - 1 \\
+ 2|\tilde{z}_3\tilde{z}_4| - 2|\tilde{z}_3\tilde{z}_5| - 1 \\
+ 2|\tilde{z}_3\tilde{z}_1| - 1 \\
+ 2|\tilde{z}_4\tilde{z}_5| - 1 \\
+ 2|\tilde{z}_4\tilde{z}_1| - 1 \\
+ 2|\tilde{z}_5\tilde{z}_2| - 1 \\
+ 2|\tilde{z}_5\tilde{z}_3| - 1 \\
+ 2|\tilde{z}_5\tilde{z}_4| - 1 \\
+ 2|\tilde{z}_5\tilde{z}_1| - 1 \\
\end{array} \right\}
\]

\[I_{C_3(D)}^* = \left\{ \begin{array}{c}
2^{|\tilde{x}\tilde{z}_1|} - 1 \\
+ 2^{|\tilde{x}\tilde{z}_2|} - 1 \\
+ 2^{|\tilde{x}\tilde{z}_3|} - 1 \\
+ 2^{|\tilde{x}\tilde{z}_4|} - 1 \\
+ 2^{|\tilde{x}\tilde{z}_5|} - 1 \\
\end{array} \right\}
\]

\[I_{C_4(D)}^* = \left\{ \begin{array}{c}
2^{|\tilde{x}\tilde{z}_1|} - 1 \\
+ 2^{|\tilde{x}\tilde{z}_2|} - 1 \\
+ 2^{|\tilde{x}\tilde{z}_3|} - 1 \\
+ 2^{|\tilde{x}\tilde{z}_4|} - 1 \\
+ 2^{|\tilde{x}\tilde{z}_5|} - 1 \\
\end{array} \right\}
\]

(see Theorems 2.11, 2.15 and 2.18).

If \(X = \{1, 2, 3, 4, 5\}\),

\[D = \{\{3\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\}
\]

then \(|I_{C_1(D)}^*| = 6, |I_{C_2(D)}^*| = 69, |I_{C_3(D)}^*| = 58, |I_{C_4(D)}^*| = 6, |I_{C_{10}^0(D)}^*| = 3, \)

\(|I_{C'(D)}^*| = 1, |I_{C''(D)}^*| = 4, |I_{C''(D)}^*| = 1, |I_D| = 148\).
THE CLASS $Z$–ELEMENTARY $X$–SEMLATTICE OF UNIONS

References


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