PARTIAL SUMS FOR A CERTAIN SUBCLASS OF MEROMORPHIC UNIVALENT FUNCTIONS

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Abstract. In this paper, the class $\Sigma_{\lambda}(\alpha, \beta, \gamma)$ of univalent meromorphic functions defined using the Ruscheweyh derivative in the punctured unit disk $U^*$ is introduced. We study some results concerning the partial sums of meromorphic univalent starlike functions and meromorphic univalent convex functions.

1. Introduction

Let $\Sigma$ denote the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \quad (a_k \geq 0), \quad (1.1)$$

which are regular and univalent in the punctured unit disc $U^* = \{ z : z \in \mathbb{C} \text{ and } 0 < |z| < 1 \} = U\{0\}$. Let $g \in \Sigma$, be given by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k, \quad (1.2)$$

then the Hadamard product (or convolution) of $f$ and $g$ is given by

$$(f \ast g)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k = (g \ast f)(z). \quad (1.3)$$

A function $f \in \Sigma$ is said to be meromorphically starlike of order $\alpha$ if

$$\text{Re} \left\{ -\frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U; \ 0 \leq \alpha < 1). \quad (1.4)$$
Denote by $\Sigma^*(\alpha)$ the class of all meromorphically starlike functions of order $\alpha$. A function $f \in \Sigma$ is said to be meromorphically convex of order $\alpha$ if

$$\text{Re}\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \alpha \quad (z \in U; \quad 0 \leq \alpha < 1).$$

(1.5)

Denote by $\Sigma^*_k(\alpha)$ the class of all meromorphically convex functions of order $\alpha$. We note that $f(z) \in \Sigma^*_k(\alpha) \iff -zf'(z) \in \Sigma^*(\alpha)$.

The classes $\Sigma^*(\alpha)$ and $\Sigma^*_k(\alpha)$ had been extensively studied by Pommerenke [7], Miller [6] and others.

For $\lambda > -1$, the Ruscheweyh derivative of order $\lambda$ is denoted by $D^\lambda f$ and is defined for function of the form (1.1) as follows: If

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k,$$

then

$$D^\lambda f(z) = \frac{1}{z} \frac{1}{(1 - z)^{\lambda+1}} * f(z) = z^{-1} + \sum_{k=1}^{\infty} D_k(\lambda) a_k z^k, \quad z \in U^*, \quad (1.6)$$

where

$$D_k(\lambda) = \frac{(\lambda + 1)(\lambda + 2) \ldots (\lambda + k + 1)}{(k + 1)!}. \quad (1.7)$$

For $\beta \geq 0$, $0 \leq \alpha < 1, 0 \leq \gamma < \frac{1}{2}$ and $\lambda > -1$, Atshan and Kulkarni [4] and Atshan [3] defined the class $\Sigma_\lambda(\alpha, \beta, \gamma)$ consisting of functions of the form (1.1) and satisfying the analytic criterion:

$$\text{Re}\left\{\frac{z(D^\lambda f(z))'}{(1 - \gamma)D^\lambda f(z) + \gamma z(D^\lambda f(z))'} + \alpha\right\} \geq \beta \left|\frac{z(D^\lambda f(z))'}{(1 - \gamma)D^\lambda f(z) + \gamma z(D^\lambda f(z))'} + 1\right| \quad (z \in U).$$

(1.8)

We note that:

$$\Sigma_0(\alpha, 0, 0) = \Sigma^*(\alpha) \quad (0 \leq \alpha < 1) \quad (\text{see Pommerenke [7]}).$$

Also, we note that

$$\Sigma_\lambda(\alpha, \beta, 0) = \Sigma^*_\lambda(\alpha, \beta) =$$

$$-\text{Re}\left\{\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} + \alpha\right\} \geq \beta \left|\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} + 1\right| \quad (z \in U).$$

(1.9)

For $\beta \geq 0$, $0 \leq \alpha < 1$, and $\lambda > -1$, we denote by $\Sigma^*_{k,\lambda}(\alpha, \beta)$ the subclass of $\Sigma$ consisting of functions of the form (1.1) and satisfying the analytic
criterion:
\[-\text{Re} \left\{ 1 + \frac{z(D^\lambda f(z))''}{D^\lambda f'(z)} + \alpha \right\} \geq \beta \left| 2 + \frac{z(D^\lambda f(z))''}{D^\lambda f'(z)} \right| (z \in U), \quad (1.10)\]

We note that:
\(\Sigma_{k,0}(\alpha, 0) = \Sigma^*_k(\alpha) (0 \leq \alpha < 1)\) (see Pommerenke [7]).

It is easy to observe from (1.9) and (1.10) that
\(f(z) \in \Sigma^*_k,\lambda (\beta, \alpha) \iff -zf'(z) \in \Sigma^*_\lambda (\beta, \alpha). \quad (1.11)\)

In order to prove our results for functions belonging to the class \(\Sigma^*_\lambda (\alpha, \beta, \gamma)\) we shall need the following lemma given by Atshan and Kulkarni [4].

**Lemma 1.** [4, Theorem 2.1] Let the function \(f\) be defined by (1.1). Then \(f \in \Sigma^*_\lambda (\alpha, \beta, \gamma)\) if and only if
\[\sum_{k=1}^{\infty} (1 + \gamma k - \gamma) [k(1 + \beta) + (\beta + \alpha)] D_k(\lambda) a_k \leq (1 - \alpha) (1 - 2\gamma). \quad (1.12)\]

where \(0 \leq \alpha < 1, \beta \geq 0, 0 \leq \gamma < \frac{1}{2}, \lambda > -1, \) and \(D_k(\lambda)\) is given by (1.7).

Taking \(\gamma = 0\) in Lemma 1, we obtain the following corollary.

**Corollary 1.** Let the function \(f\) defined by (1.1). Then \(f \in \Sigma^*_\lambda (\beta, \alpha)\) if and only if
\[\sum_{k=1}^{\infty} k(1 + \beta) + (\beta + \alpha)] D_k(\lambda) a_k \leq (1 - \alpha). \quad (1.13)\]

By using Corollary 1 and (1.11), we can prove the following lemma.

**Lemma 2.** Let the function \(f\) defined by (1.1). Then \(f \in \Sigma^*_{k,\lambda} (\beta, \alpha)\) if and only if
\[\sum_{k=1}^{\infty} k [k(1 + \beta) + (\beta + \alpha)] D_k(\lambda) a_k \leq (1 - \alpha). \quad (1.14)\]

In this paper, applying the technique used by Silverman [8], we will investigate the ratio of a function of the form (1.1) to its sequence of partial sums \(f_n(z) = \frac{1}{2} + \sum_{k=1}^{n} a_k z^k\) when the coefficients of \(f\) are sufficiently small to satisfy condition (1.13) or (1.14). More precisely, we will determine sharp lower bounds for
\[\text{Re} \left\{ \frac{f(z)}{f_n(z)} \right\}, \text{Re} \left\{ \frac{f_n(z)}{f(z)} \right\}, \text{Re} \left\{ \frac{f'(z)}{f'(z)} \right\}, \text{Re} \left\{ \frac{f'(z)}{f'(z)} \right\} .\]

In the sequel, we will make use of well-known result that \(\text{Re} \left\{ \frac{1+w(z)}{1-w(z)} \right\} > 0 (z \in U)\) if and only if \(w(z) = \sum_{k=1}^{\infty} c_k z^k\) satisfies the inequality \(|w(z)| \leq |z|\).
Unless otherwise stated, we will assume that \( f \) is of the form (1.1) and its sequence of partial sums is denoted by \( f_n(z) = \frac{1}{z} + \sum_{k=1}^{n} a_k z^{-k} \).

For the notational convenience we shall henceforth denote
\[
\delta_k(\lambda, \beta, \alpha) = [k(1 + \beta) + (\beta + \alpha)] D_k(\lambda). \quad (1.15)
\]

\section{Main results}

\bf{Theorem 1.} If \( f \) of the form (1.1) satisfies condition (1.13), then
\[
\text{Re} \left\{ \frac{f(z)}{f_n(z)} \right\} \geq \frac{\delta_{n+1}(\lambda, \beta, \alpha) - (1 - \alpha)}{\delta_{n+1}(\lambda, \beta, \alpha)} \quad (z \in U). \quad (2.1)
\]

The result is sharp, with extremal function
\[
f(z) = \frac{1}{z} + \frac{1 - \alpha}{\delta_{n+1}(\lambda, \beta, \alpha)} z^{n+1} \quad (n \geq 1). \quad (2.2)
\]

\bf{Proof.} We may write
\[
\frac{\delta_{n+1}(\lambda, \beta, \alpha)}{1 - \alpha} \left[ \frac{f(z)}{f_n(z)} - \frac{\delta_{n+1}(\lambda, \beta, \alpha) - (1 - \alpha)}{\delta_{n+1}(\lambda, \beta, \alpha)} \right] = \frac{1 + \sum_{k=1}^{n} a_k z^{-k+1} + \frac{\delta_{n+1}(\lambda, \beta, \alpha)}{1 - \alpha} \sum_{k=n+1}^{\infty} a_k z^{-k+1}}{1 + \sum_{k=1}^{n} a_k z^{-k+1}} = 1 + A(z) \]
\[
= \frac{1}{1 + B(z)},
\]

Set \( \frac{1 + A(z)}{1 + B(z)} = \frac{1 + w(z)}{1 - w(z)} \), so that \( w(z) = \frac{A(z) - B(z)}{2 + A(z) + B(z)} \). Then
\[
w(z) = \frac{\delta_{n+1}(\lambda, \beta, \alpha)}{1 - \alpha} \sum_{k=n+1}^{\infty} a_k z^{-k+1}
\]
\[
= \frac{2 + 2 \sum_{k=1}^{n} a_k z^{-k+1} + \delta_{n+1}(\lambda, \beta, \alpha)}{2 + 2 \sum_{k=1}^{n} a_k - \delta_{n+1}(\lambda, \beta, \alpha)} \sum_{k=n+1}^{\infty} a_k z^{-k+1}
\]

and
\[
|w(z)| \leq \frac{\delta_{n+1}(\lambda, \beta, \alpha)}{1 - \alpha} \sum_{k=n+1}^{\infty} a_k \]
\[
= 2 - 2 \sum_{k=1}^{n} a_k - \frac{\delta_{n+1}(\lambda, \beta, \alpha)}{1 - \alpha} \sum_{k=n+1}^{\infty} a_k.
\]

Now \( |w(z)| \leq 1 \) if and only if
\[
2 \left( \frac{\delta_{n+1}(\lambda, \beta, \alpha)}{1 - \alpha} \right) \sum_{k=n+1}^{\infty} a_k \leq 2 - 2 \sum_{k=1}^{n} a_k,
\]
which is equivalent to
\[
\sum_{k=1}^{n} a_k + \left( \frac{\delta_{n+1}(\lambda, \beta, \alpha)}{1-\alpha} \right) \sum_{k=n+1}^{\infty} a_k \leq 1. \tag{2.3}
\]

It suffices to show that the left hand side of (2.3) is bounded above by
\[
\sum_{k=1}^{\infty} \left( \frac{\delta_k(\lambda, \beta, \alpha)}{1-\alpha} \right) a_k,
\]
which is equivalent to
\[
\sum_{k=1}^{n} \frac{\delta_k(\lambda, \beta, \alpha) - (1-\alpha)}{1-\alpha} a_k + \sum_{k=n+1}^{\infty} \frac{\delta_k(\lambda, \beta, \alpha) - \delta_{n+1}(\lambda, \beta, \alpha)}{1-\alpha} a_k \geq 0.
\]

To see that the function \( f \) given by (2.2) gives the sharp result, we observe for \( z = re^{\pi i/(n+2)} \) that
\[
\frac{f(z)}{f_n(z)} = 1 + \frac{1-\alpha}{\delta_{n+1}(\lambda, \beta, \alpha)} z^{n+2} = 1 - \frac{1-\alpha}{\delta_{n+1}(\lambda, \beta, \alpha)} = \frac{\delta_{n+1}(\lambda, \beta, \alpha) - (1-\alpha)}{\delta_{n+1}(\lambda, \beta, \alpha)} \text{ when } r \to 1^{-}.
\]

Therefore the proof of Theorem 1 is completed. \( \square \)

**Theorem 2.** If \( f \) of the form (1.1) satisfies condition (1.14), then
\[
\operatorname{Re} \left\{ \frac{f(z)}{f_n(z)} \right\} \geq \frac{(n+1)\delta_{n+1}(\lambda, \beta, \alpha) - (1-\alpha)}{(n+1)\delta_{n+1}(\lambda, \beta, \alpha)} \quad (z \in U). \tag{2.4}
\]

The result is sharp for every \( n \), with extremal function
\[
f(z) = \frac{1}{z} + \frac{1-\alpha}{(n+1)\delta_{n+1}(\lambda, \beta, \alpha)} z^{n+1} \quad (n \geq 1). \tag{2.5}
\]

**Proof.** We may write
\[
\frac{(n+1)\delta_{n+1}(\lambda, \beta, \alpha)}{1-\alpha} \left[ \frac{f(z)}{f_n(z)} - \frac{(n+1)\delta_{n+1}(\lambda, \beta, \alpha) - (1-\alpha)}{(n+1)\delta_{n+1}(\lambda, \beta, \alpha)} \right]
\]

\[=\frac{1 + \sum_{k=1}^{n} a_k z^{k+1} + \frac{(n+1)\delta_{n+1}(\lambda, \beta, \alpha)}{1-\alpha} \sum_{k=n+1}^{\infty} a_k z^{k+1}}{1 + \sum_{k=1}^{n} a_k z^{k+1}}
\]

\[=\frac{1 + w(z)}{1 - w(z)}
\]

which implies
\[
\sum_{k=1}^{n} a_k \leq \frac{(n+1)\delta_{n+1}(\lambda, \beta, \alpha)}{1-\alpha} \sum_{k=n+1}^{\infty} a_k.
\]
where
\[ w(z) = \frac{(n+1)\delta_{n+1}(\lambda, \beta, \alpha)}{1-\alpha} \sum_{k=n+1}^{\infty} a_k z^{k+1} + 2 + 2 \sum_{k=1}^{n} a_k z^{k+1} + \frac{(n+1)\delta_{n+1}(\lambda, \beta, \alpha)}{1-\alpha} \sum_{k=n+1}^{\infty} a_k z^{k+1}. \]

Now
\[ |w(z)| \leq \frac{(n+1)\delta_{n+1}(\lambda, \beta, \alpha)}{1-\alpha} \sum_{k=n+1}^{\infty} a_k, \]
if
\[ \sum_{k=1}^{n} a_k + \frac{(n+1)\delta_{n+1}(\lambda, \beta, \alpha)}{1-\alpha} \sum_{k=n+1}^{\infty} a_k \leq 1. \quad (2.6) \]

The left hand side of (2.6) is bounded above by
\[ \sum_{k=1}^{\infty} \left( \frac{k\delta_k(\lambda, \beta, \alpha)}{1-\alpha} \right) a_k, \]
if
\[ \frac{1}{1-\alpha} \left\{ \sum_{k=1}^{n} [k\delta_k(\lambda, \beta, \alpha) - (1-\alpha)] a_k + \sum_{k=n+1}^{\infty} [k\delta_k(\lambda, \beta, \alpha) - (n+1)\delta_{n+1}(\lambda, \beta, \alpha)] a_k \right\} \geq 0 \]
and the proof of Theorem 2 is completed.

We next determine bounds for Re\( \left\{ \frac{f_n(z)}{f(z)} \right\} \).

**Theorem 3.** (a) If \( f \) of the form (1.1) satisfies condition (1.13), then
\[ \text{Re} \left\{ \frac{f_n(z)}{f(z)} \right\} \geq \frac{\delta_{n+1}(\lambda, \beta, \alpha)}{(1-\alpha) + \delta_{n+1}(\lambda, \beta, \alpha)} \quad (z \in U). \quad (2.7) \]

(b) If \( f \) of the form (1.1) satisfies condition (1.14), then
\[ \text{Re} \left\{ \frac{f_n(z)}{f(z)} \right\} \geq \frac{(n+1)\delta_{n+1}(\lambda, \beta, \alpha)}{(1-\alpha) - (n+1)\delta_{n+1}(\lambda, \beta, \alpha)} \quad (z \in U). \quad (2.8) \]

Equalities hold in (a) and (b) for the functions given by (2.2) and (2.5), respectively.
Proof. We prove $(a)$. The proof of $(b)$ is similar to $(a)$ and will be omitted. We write

$$
\frac{(1 - \alpha) + \delta_{n+1}(\lambda, \beta, \alpha)}{1 - \alpha} \left[ \frac{f_n(z)}{f(z)} - \frac{\delta_{n+1}(\lambda, \beta, \alpha)}{(1 - \alpha) + \delta_{n+1}(\lambda, \beta, \alpha)} \right] = \frac{1 + \sum_{k=1}^{n} a_k z^{k+1} - \frac{\delta_{n+1}(\lambda, \beta, \alpha)}{1 - \alpha} \sum_{k=n+1}^{\infty} a_k z^{k+1}}{1 + \sum_{k=1}^{\infty} a_k z^{k+1}} = \frac{1 + w(z)}{1 - w(z)},
$$

where

$$
|w(z)| \leq \frac{(1 - \alpha) + \delta_{n+1}(\lambda, \beta, \alpha)}{1 - \alpha} \sum_{k=n+1}^{\infty} a_k \leq 1.
$$

This last inequality is equivalent to

$$
\sum_{k=1}^{n} a_k + \left( \frac{\delta_{n+1}(\lambda, \beta, \alpha)}{1 - \alpha} \right) \sum_{k=n+1}^{\infty} a_k \leq 1. \quad (2.9)
$$

Since the left hand side of (2.9) is bounded above by

$$
\sum_{k=1}^{\infty} \left( \frac{\delta_k(\lambda, \beta, \alpha)}{1 - \alpha} \right) a_k,
$$

the proof is completed. \(\square\)

We next turn to ratios involving derivatives. The proof of Theorem 4 below follows the pattern of those in Theorem 1 and $(a)$ of Theorem 3 and so the details may be omitted.

**Theorem 4.** If $f$ of the form (1.1) satisfies condition (1.13), then

$$
(a) \Re \left\{ \frac{f'(z)}{f_n(z)} \right\} \geq \frac{\delta_{n+1}(\lambda, \beta, \alpha) + (n + 1)(1 - \alpha)}{\delta_{n+1}(\lambda, \beta, \alpha)} \quad (z \in U),
$$

$$
(b) \Re \left\{ \frac{f'_n(z)}{f'(z)} \right\} \geq \frac{\delta_{n+1}(\lambda, \beta, \alpha)}{\delta_{n+1}(\lambda, \beta, \alpha) - (n + 1)(1 - \alpha)} \quad (z \in U; \alpha \neq 0). \quad \text{The extremal function for the case (a) is given by (2.2) and the extremal function for the case (b) is given by (2.2) with \(\alpha \neq 0\).}
Remark 1. Putting $\beta = 0$ and $\lambda = 0$ in Theorem 2, we obtain the following corollary:

**Corollary 2.** If $f$ of the form (1.1) satisfies condition (1.13) (with $\beta = 0$ and $\lambda = 0$), that is $f \in \Sigma^*(\alpha)$, then

(a) $\text{Re} \left\{ \frac{f'(z)}{f_n'(z)} \right\} \geq \frac{2(n+1)-n\alpha}{n+1+\alpha} \quad (z \in U),$

(b) $\text{Re} \left\{ \frac{f_n'(z)}{f'(z)} \right\} \geq \frac{n+1+\alpha}{\alpha(n+2)} \quad (z \in U; \alpha \neq 0).$

The extremal function for the case (a) is given by (2.2) (with $\beta = 0$ and $\lambda = 0$) and the extremal function for the case (b) is given by (2.2). (with $\beta = 0$, $\lambda = 0$ and $\alpha \neq 0$).

Remark 2. We note that Corollary 2 corrects the result obtained by Cho and Owa [5, Theorem 4].

**Theorem 5.** If $f$ of the form (1.1) satisfies condition (1.14), then

(a) $\text{Re} \left\{ \frac{f'(z)}{f_n'(z)} \right\} \geq \frac{\delta_{n+1}(\lambda, \beta, \alpha) - (1-\alpha)}{\delta_{n+1}(\lambda, \beta, \alpha)} \quad (z \in U).$

(b) $\text{Re} \left\{ \frac{f_n'(z)}{f'(z)} \right\} \geq \frac{\delta_{n+1}(\lambda, \beta, \alpha)}{(1-\alpha) + \delta_{n+1}(\lambda, \beta, \alpha)} \quad (z \in U).$

In both cases, the extremal function is given by (2.5).

**Proof.** It is well known that $f \in \Sigma^*_k(\alpha) \iff -zf' \in \Sigma^*(\alpha)$. In particular, $f$ satisfies condition (1.14) if and only if $-zf'$ satisfies condition (1.13). Thus, (a) is an immediate consequence of Theorem 1 and (b) follows directly from (a) of Theorem 3. \qed

Remark 3. Putting $\beta = 0$ and $\lambda = 0$, in the above results, we get the results obtained by Cho and Owa [5].

Remark 4. Putting $\beta = 0$ and $\lambda = 0$, in the above results, we get the results obtained by Aouf and Silverman [2 with $p = 1$].

Remark 5. Putting $\beta = 0$ and $\lambda = 0$, in the above results, we get the results obtained by Aouf and Mostafa [1 with $p = B = 1$ and $A = 2\alpha - 1$, $0 \leq \alpha < 1$].
References


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