SUPERADDITIVITY OF FUNCTIONALS RELATED TO
GAUSS’ TYPE INEQUALITIES

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Abstract. In this paper we prove superadditivity of some functionals associated with the Gauss-Winckler and the Gauss-Pólya inequalities.

1. Introduction

In [2] C. F. Gauss mentioned the following inequality between the second and the fourth absolute moments.

If \( f \) is a non-negative and decreasing function, then

\[
\left( \int_0^\infty x^2 f(x) \, dx \right)^2 \leq \frac{5}{9} \int_0^\infty f(x) \, dx \int_0^\infty x^4 f(x) \, dx.
\]

(1.1)

Until now, there are a lot of generalizations, sharpenings and improvements of inequality (1.1). One of major lines of generalization is due to A. Winckler and the other springing from a pair of results of G. Pólya.

A. Winckler, [7], gave the following result which is known as the Gauss-Winckler inequality in the recent literature. More about it and its history one can find in [1].

Theorem 1.1. If \( f \) is a non-negative, continuous and non-increasing function on \([0, \infty)\) such that \( \int_0^\infty f(x) \, dx = 1 \), then for \( m \leq r \)

\[
\left( (m+1) \int_0^\infty x^m f(x) \, dx \right)^{\frac{1}{m}} \leq \left( (r+1) \int_0^\infty x^r f(x) \, dx \right)^{\frac{1}{r}}.
\]

(1.2)

Another generalization was done by G. Pólya and today those type of inequalities are called the Gauss-Pólya inequalities. Namely, in the book "Problems and Theorems in Analysis" (see [5, Vol I, p. 83, Vol II, p. 129] one can find the following results.

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Theorem 1.2.

(i) Let \( f : [0, \infty) \to \mathbb{R} \) be a non-negative and decreasing function. If \( a \) and \( b \) are non-negative real numbers, then
\[
\left( \int_0^\infty x^{a+b} f(x) \, dx \right)^2 \leq \left( 1 - \left( \frac{a-b}{a+b+1} \right)^2 \right) \int_0^\infty x^{2a} f(x) \, dx \int_0^\infty x^{2b} f(x) \, dx
\]
if all the integrals exist.

(ii) Let \( f : [0, 1] \to \mathbb{R} \) be a non-negative and increasing function. If \( a \) and \( b \) are non-negative real numbers, then
\[
\left( \int_0^1 x^{a+b} f(x) \, dx \right)^2 \geq \left( 1 - \left( \frac{a-b}{a+b+1} \right)^2 \right) \int_0^1 x^{2a} f(x) \, dx \int_0^1 x^{2b} f(x) \, dx.
\]

J. Pečarić and S. Varošanec treated the above mentioned inequalities in a unified way and proved the following generalizations, [4], [6].

Theorem 1.3. Let \( g : [a, b] \to \mathbb{R} \) be a non-negative increasing differentiable function and let \( f : [a, b] \to \mathbb{R} \), be a non-negative function such that \( x \mapsto f(x)g'(x) \) is a non-decreasing function. Let \( p_i (i = 1, \ldots, n) \) be positive real numbers such that \( \sum_{i=1}^n \frac{1}{p_i} = 1 \). If \( a_i (i = 1, \ldots, n) \) are real numbers such that \( a_i > -\frac{1}{p_i} \), then
\[
\int_a^b g(x)^{a_1 + \cdots + a_n} f(x) \, dx \geq \prod_{i=1}^n (a_i p_i + 1) \frac{1}{1 + \sum_{i=1}^n a_i} \prod_{i=1}^n \left( \int_a^b g(x)^{a_i p_i} f(x) \, dx \right)^{\frac{1}{p_i}}.
\]
(1.3)

If \( g(a) = 0 \) and if the quotient function \( \frac{f}{g} \) is non-increasing, then the reverse inequality in (1.3) holds.

As a consequence of the above results we conclude that if \( f \) and \( g \) satisfy the assumptions of Theorem 1.3, then the function
\[
Q(r) = (r + 1) \int_a^b g^*(x) f(x) \, dx
\]
is log-concave when \( \frac{f}{g} \) is a non-decreasing function and the function \( Q \) is log-convex when \( g(a) = 0 \) and \( \frac{f}{g} \) is non-increasing.

Using that property, the following generalization of the Gauss-Winckler inequality was proved in [6]:

Theorem 1.4. Let \( f \) and \( g \) be defined as in Theorem 1.3, \( \frac{f}{g} \) be a non-decreasing function and \( p, q, r, s \) be real numbers from the domain of definition of the function \( Q \).
If \( p \leq q, r \leq s \) and \( p > r, q > s \), then
\[
\left( \frac{(p+1) \int_a^b g^p(x)f(x)\,dx}{(r+1) \int_a^b g^r(x)f(x)\,dx} \right)^{\frac{1}{p-r}} \geq \left( \frac{(q+1) \int_a^b g^q(x)f(x)\,dx}{(s+1) \int_a^b g^s(x)f(x)\,dx} \right)^{\frac{1}{q-s}}. \tag{1.4}
\]

If \( g(a) = 0 \) and \( \frac{g'}{g} \) is non-increasing, then the reverse inequality holds.

**Remark 1.5.** In [6] the authors considered the case when \( g(x) = x \), \( f \) is non-increasing and \( a = 0 \). In that case inequalities (1.3) and (1.4) hold with \( b = \infty \) and then we get results for moments.

In the next section we investigate properties of mappings which arise from Gauss-Pólya’s inequalities, while in the third section we research functional related to the Gauss-Winckler inequality (1.4). The main tool of this investigation is the Hölder type inequality which we give in the following form, [3]:

**Proposition 1.6.** Let \( a_i, b_i, p_i, (i = 1, \ldots, n) \) be non-negative real numbers such that \( \sum_{i=1}^n \frac{1}{p_i} = 1 \). Then
\[
a_1^{\frac{1}{p_1}} \cdots a_n^{\frac{1}{p_n}} + b_1^{\frac{1}{p_1}} \cdots b_n^{\frac{1}{p_n}} \leq \prod_{i=1}^n (a_i + b_i)^{\frac{1}{p_i}}. \tag{1.5}
\]

It is a simple consequence of weighted AM-GM inequality
\[
\frac{a_1^{\frac{1}{p_1}} \cdots a_n^{\frac{1}{p_n}}}{(a_1 + b_1)^{\frac{1}{p_1}} \cdots (a_n + b_n)^{\frac{1}{p_n}}} + \frac{b_1^{\frac{1}{p_1}} \cdots b_n^{\frac{1}{p_n}}}{(a_1 + b_1)^{\frac{1}{p_1}} \cdots (a_n + b_n)^{\frac{1}{p_n}}}
\leq \frac{a_1}{p_1(a_1 + b_1)} + \cdots + \frac{a_n}{p_n(a_n + b_n)} + \frac{b_1}{p_1(a_1 + b_1)} + \cdots + \frac{b_n}{p_n(a_n + b_n)} = 1.
\]

2. **Functionals related to the Gauss-Pólya inequalities**

Throughout this section functions \( f, g : [a, b] \to \mathbb{R} \) are non-negative, \( g \) is increasing differentiable, numbers \( p_i (i = 1, \ldots, n) \) are positive reals such that \( \sum_{i=1}^n \frac{1}{p_i} = 1 \) and \( a_i (i = 1, \ldots, n) \) are real numbers such that \( a_i > -\frac{1}{p_i} \).

Let us consider the functional \( G \) defined as
\[
G(f) = \prod_{i=1}^n (a_i p_i + 1)^{\frac{1}{p_i}} \prod_{i=1}^n \left( \int_a^b g(x)^{a_i p_i} f(x) \, dx \right)^{\frac{1}{p_i}} - (1 + \sum_{i=1}^n a_i) \int_a^b g(x)^{a_1 + \cdots + a_n} f(x) \, dx.
\]
It is obvious that \( f \mapsto G(f) \) is positive homogeneous, i.e. \( G(\lambda f) = \lambda G(f) \) for any \( \lambda \geq 0 \). As a consequence of Theorem 1.3, if \( f/g' \) is a non-decreasing function, then \( G(f) \leq 0 \), while if \( f/g' \) is non-increasing and \( g(a) = 0 \), then \( G(f) \geq 0 \).

The following theorem gives superadditivity property of the functional \( G \).

**Theorem 2.1.** Let \( f_1, f_2, g : [a, b] \to \mathbb{R} \) be non-negative functions, \( g \) increasing differentiable, numbers \( p_i (i = 1, \ldots, n) \) be positive reals such that \( \sum_{i=1}^{n} \frac{1}{p_i} = 1 \) and \( a_i (i = 1, \ldots, n) \) be real numbers such that \( a_i > -\frac{1}{p_i} \). Then

\[
G(f_1 + f_2) \geq G(f_1) + G(f_2),
\]

i.e. \( G \) is a superadditive functional.

Furthermore, if \( f_1 \geq f_2 \) such that \( \frac{f_1 - f_2}{g'} \) is non-increasing, \( g(a) = 0 \), then

\[
G(f_1) \geq G(f_2),
\]

i.e. \( G \) is non-decreasing.

**Proof.** Let us consider a difference \( G(f_1 + f_2) - G(f_1) - G(f_2) \).

\[
G(f_1 + f_2) - G(f_1) - G(f_2) = \prod_{i=1}^{n} (a_i p_i + 1) \prod_{i=1}^{n} \left( \int_{a}^{b} g(x)^{a_i p_i} (f_1 + f_2)(x) \, dx \right)^{\frac{1}{p_i}}
- \left( 1 + \sum_{i=1}^{n} a_i \right) \int_{a}^{b} g(x)^{a_1 + \cdots + a_n} (f_1 + f_2)(x) \, dx
- \prod_{i=1}^{n} (a_i p_i + 1) \prod_{i=1}^{n} \left( \int_{a}^{b} g(x)^{a_i p_i} f_1(x) \, dx \right)^{\frac{1}{p_i}}
+ (1 + \sum_{i=1}^{n} a_i) \int_{a}^{b} g(x)^{a_1 + \cdots + a_n} f_1(x) \, dx
- \prod_{i=1}^{n} (a_i p_i + 1) \prod_{i=1}^{n} \left( \int_{a}^{b} g(x)^{a_i p_i} f_2(x) \, dx \right)^{\frac{1}{p_i}}
+ (1 + \sum_{i=1}^{n} a_i) \int_{a}^{b} g(x)^{a_1 + \cdots + a_n} f_2(x) \, dx
= \prod_{i=1}^{n} (a_i p_i + 1) \prod_{i=1}^{n} \left( \int_{a}^{b} g(x)^{a_i p_i} (f_1 + f_2)(x) \, dx \right)^{\frac{1}{p_i}}
- \prod_{i=1}^{n} \left( \int_{a}^{b} g(x)^{a_i p_i} f_1(x) \, dx \right)^{\frac{1}{p_i}}
- \prod_{i=1}^{n} \left( \int_{a}^{b} g(x)^{a_i p_i} f_2(x) \, dx \right)^{\frac{1}{p_i}}.
\]
are non-negative non-increasing functions. Then
\[ c, C \in \mathbb{R} \]
and using the Hölder inequality we have that \( G(f_1 + f_2) - G(f_1) - G(f_2) \geq 0 \), so \( G \) is superadditive.

If \( f_1 \geq f_2 \), \( \frac{f_1 - f_2}{g} \) is non-increasing and \( g(a) = 0 \), then \( G(f_1 - f_2) \geq 0 \), so, we have
\[
G(f_1) = G(f_2 + (f_1 - f_2)) \\
\geq G(f_2) + G(f_1 - f_2) \geq G(f_2).
\]

\[ \square \]

**Corollary 2.2.** Let \( f_1, f_2, g \) be non-negative functions on \([a, b]\), \( g \) increasing differentiable, \( g(a) = 0 \), numbers \( p_i (i = 1, \ldots, n) \) be positive reals such that \( \sum_{i=1}^{n} \frac{1}{p_i} = 1 \), \( a_i (i = 1, \ldots, n) \) be real numbers such that \( a_i > -\frac{1}{p_i} \) and \( c, C \in \mathbb{R} \) such that \( C f_2 - f_1 \), \( f_1 - c f_2 \) are non-negative and \( \frac{C f_2 - f_1}{g} \), \( \frac{f_1 - c f_2}{g} \) are non-negative non-increasing functions. Then
\[
C\left\{ \prod_{i=1}^{n} (a_ip_i + 1)^\frac{1}{p_i} \prod_{i=1}^{n} \left( \int_{a}^{b} g(x)^{a_ip_i} f_2(x) \, dx \right)^\frac{1}{p_i} \right\} \\
\geq \prod_{i=1}^{n} (a_ip_i + 1)^\frac{1}{p_i} \prod_{i=1}^{n} \left( \int_{a}^{b} g(x)^{a_ip_i} f_1(x) \, dx \right)^\frac{1}{p_i} \\
\geq C\left\{ \prod_{i=1}^{n} (a_ip_i + 1)^\frac{1}{p_i} \prod_{i=1}^{n} \left( \int_{a}^{b} g(x)^{a_ip_i} f_2(x) \, dx \right)^\frac{1}{p_i} \right\}
\]

\[ -(1 + \sum_{i=1}^{n} a_i) \int_{a}^{b} g(x)^{a_1 + \ldots + a_n} f_2(x) \, dx \]

**Proof.** Using previous results we have
\[
CG(f_2) = G(Cf_2) = G((C f_2 - f_1) + f_1) \geq G(C f_2 - f_1) + G(f_1) \geq G(f_1)
\]
and
\[
G(f_1) = G((f_1 - c f_2) + c f_2) \geq G(f_1 - c f_2) + G(c f_2) \geq G(c f_2) = c G(f_2)
\]
from which the conclusion of the corollary is established.

The following theorem contains a result about concavity of function $G \circ \phi$ where $\phi$ is concave.

**Theorem 2.3.** Let $\phi : [0, \infty) \to [0, \infty)$ be a concave function, $f_1$, $f_2$, $g$ be non-negative functions on $[a, b]$ such that $(\phi \circ (\alpha f_1 + (1 - \alpha) f_2) - [\alpha(\phi \circ f_1) + (1 - \alpha)(\phi \circ f_2)])/g'$ is non-increasing for some $\alpha \in [0, 1]$, $g(a) = 0$. Then

$$G \circ \phi \circ (\alpha f_1 + (1 - \alpha) f_2) \geq \alpha (G \circ \phi \circ f_1) + (1 - \alpha)(G \circ \phi \circ f_2).$$

**Proof.** For any $x \in [a, b]$ we have

$$(\phi \circ (\alpha f_1 + (1 - \alpha) f_2))(x) = \phi(\alpha f_1(x) + (1 - \alpha) f_2(x))$$

$$\geq \alpha \phi(f_1(x)) + (1 - \alpha)\phi(f_2(x))$$

$$= (\alpha(\phi \circ f_1) + (1 - \alpha)(\phi \circ f_2))(x),$$

where a concavity of function $\phi$ is used. So, we have $\phi \circ (\alpha f_1 + (1 - \alpha) f_2) \geq \alpha(\phi \circ f_1) + (1 - \alpha)(\phi \circ f_2)$. Using properties of $G$ and the above-proved inequality we have

$$G(\phi \circ (\alpha f_1 + (1 - \alpha) f_2)) \geq G(\alpha(\phi \circ f_1) + (1 - \alpha)(\phi \circ f_2))$$

$$\geq G(\alpha(\phi \circ f_1)) + G((1 - \alpha)(\phi \circ f_2)) = \alpha G(\phi \circ f_1) + (1 - \alpha)G(\phi \circ f_2)$$

and the proof is established.

**Remark 2.4.** Let us consider a case when $g(x) = x$, $a = 0$, $b = \infty$ and $f$ is non-increasing as it is mentioned in Remark 1.5. Let us denote by $\mu_r(f)$ a moment of the order $r$ i.e.

$$\mu_r(f) = \int_0^\infty x^r f(x) \, dx.$$ 

Then the functional $G$ has a form

$$G(f) = \prod_{i=1}^n (a_i p_i + 1)^{\frac{1}{p}} \prod_{i=1}^n \mu_{\alpha_i p_i}(f) - (1 + \sum_{i=1}^n a_i)\mu_{a_1 + \cdots + a_n}(f)$$

and $G$ is superadditive. Also, if $f_1 \geq f_2$ such that $f_1 - f_2$ is non-increasing, then $G(f_1) \geq G(f_2)$.

3. **Functionals related to the Gauss-Winckler inequality**

Putting in (1.4) $r = s = 0$ we get the Gauss-Winckler inequality for $f/g'$ non-decreasing function:

$$\left( \frac{\int_a^b g^p(x) f(x) \, dx}{\int_a^b f(x) \, dx} \right)^{\frac{1}{p}} \geq \left( \frac{\int_a^b g^q(x) f(x) \, dx}{\int_a^b f(x) \, dx} \right)^{\frac{1}{q}}.$$
where $0 < p \leq q$. If $f/g'$ is non-increasing and $g(a) = 0$, then the reversed inequality holds.

Let us consider a functional $W$ defined as

$$W(f) = \left( \int_a^b f(x) \, dx \right)^{1 - \frac{p}{q}} \left( (q + 1) \int_a^b g^q(x) f(x) \, dx \right)^{\frac{p}{q}} - (p + 1) \int_a^b g^p(x) f(x) \, dx.$$ 

The following theorem gives superadditivity and monotonicity of the functional $W$.

**Theorem 3.1.** Let $f_1, f_2, g : [a, b] \to \mathbb{R}$ be non-negative functions, $g$ increasing differentiable, numbers $p, q$ be positive real such that $p \leq q$. Then

$$W(f_1 + f_2) \geq W(f_1) + W(f_2).$$

Additionally, if $f_1 \geq f_2$ such that $\frac{f_1 - f_2}{g'}$ is non-increasing, $g(a) = 0$, then

$$W(f_1) \geq W(f_2).$$

**Proof.** Let us transform $W(f_1 + f_2) - W(f_1) - W(f_2)$.

$$W(f_1 + f_2) - W(f_1) - W(f_2)$$

$$= \left( \int_a^b (f_1 + f_2)(x) \, dx \right)^{1 - \frac{p}{q}} \left( (q + 1) \int_a^b g^q(x)(f_1 + f_2)(x) \, dx \right)^{\frac{p}{q}}$$

$$- (p + 1) \int_a^b g^p(x)(f_1 + f_2)(x) \, dx$$

$$- \left( \int_a^b f_1(x) \, dx \right)^{1 - \frac{p}{q}} \left( (q + 1) \int_a^b g^q(x)f_1(x) \, dx \right)^{\frac{p}{q}}$$

$$+ (p + 1) \int_a^b g^p(x)f_1(x) \, dx - \left( \int_a^b f_2(x) \, dx \right)^{1 - \frac{p}{q}} \left( (q + 1) \int_a^b g^q(x)f_2(x) \, dx \right)^{\frac{p}{q}}$$

$$+ (p + 1) \int_a^b g^p(x)f_2(x) \, dx$$

$$= \left( \int_a^b (f_1 + f_2)(x) \, dx \right)^{1 - \frac{p}{q}} \left( (q + 1) \int_a^b g^q(x)(f_1 + f_2)(x) \, dx \right)^{\frac{p}{q}}$$

$$- \left( \int_a^b f_1(x) \, dx \right)^{1 - \frac{p}{q}} \left( (q + 1) \int_a^b g^q(x)f_1(x) \, dx \right)^{\frac{p}{q}}$$

$$- \left( \int_a^b f_2(x) \, dx \right)^{1 - \frac{p}{q}} \left( (q + 1) \int_a^b g^q(x)f_2(x) \, dx \right)^{\frac{p}{q}} \geq 0.$$
where in the last inequality we use the Hölder inequality with
\[ n = 2, \quad \frac{1}{p_1} = 1 - \frac{P}{q} > 0, \quad \frac{1}{p_2} = \frac{P}{q} > 0, \quad a_1 = \int_a^b f_1(x) \, dx, \quad b_1 = \int_a^b f_2(x) \, dx, \]
\[ a_2 = (q + 1) \int_a^b g^q(x)f_1(x) \, dx, \quad b_2 = (q + 1) \int_a^b g^q(x)f_2(x) \, dx. \]
So, superadditivity of the functional \( W \) is established.

If \( \frac{f_1 - f_2}{g^q} \) is non-increasing, \( g(a) = 0 \), then from Theorem 1.4 we obtain
\[ W(f_1 - f_2) \geq 0 \quad \text{and} \quad W(f_1) = W(f_2 + (f_1 - f_2)) \geq W(f_2) + W(f_1 - f_2) \geq W(f_2). \]

\[ \square \]

**Remark 3.2.** Let us consider a case when \( g(x) = x, \ a = 0, \ b = \infty \) and \( f \) is non-increasing as it is mentioned in Remark 1.5. Now the functional \( W \) has the form
\[ W(f) = (q + 1)^\frac{p}{q} \left( \mu_0(f) \right)^{1 - \frac{p}{q}} \mu_\frac{p}{q}(f) - (p + 1)\mu_p(f) \]
and \( W \) is superadditive. Also, if \( f_1 \geq f_2 \) such that \( f_1 - f_2 \) is non-increasing, then \( W(f_1) \geq W(f_2) \).

The following result is an interesting inequality for the Beta function.

**Corollary 3.3.** Let \( 0 < p \leq q, \ y_1, y_2 > -1. \) Then
\[ \left( \frac{1}{y_1 + 1} + \frac{1}{y_2 + 1} \right)^{1 - \frac{p}{q}} \left[ \beta(q + 1, y_1 + 1) + \beta(q + 1, y_2 + 1) \right]^{\frac{p}{q}} \]
\[ \geq \left( \frac{1}{y_1 + 1} \right)^{1 - \frac{p}{q}} \beta^{\frac{p}{q}}(q + 1, y_1 + 1) + \left( \frac{1}{y_2 + 1} \right)^{1 - \frac{p}{q}} \beta^{\frac{p}{q}}(q + 1, y_2 + 1) \]
where \( \beta \) is the Beta function defined as \( \beta(x + 1, y + 1) = \int_0^1 t^x(1 - t)^y \, dt. \)

**Proof.** It is a consequence of the previous theorem with \( [a, b] = [0, 1], \) \( f_i(t) = (1 - t)^{y_i}, \ i = 1, 2, \ g(x) = x. \)

\[ \square \]

**References**


