ON HYERS-ULAM STABILITY OF WILSON’S FUNCTIONAL EQUATION ON $P_3$-GROUPS

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Abstract. The purposes of paper is to obtain the Hyers-Ulam stability of Wilson’s equation
$\varphi(xy) + \varphi(xy^{-1}) = 2\varphi(x)\varphi(y)$ for $\varphi, \phi : G \to K$, where $G$ is
a $P_3$-group and $K$ a field with char$K \neq 2$.

1. Introduction

In 1989, Aczél, Chung and Ng have solved Wilson’s equation,
$\varphi(xy) + \varphi(xy^{-1}) = 2\varphi(x)\varphi(y)$
assuming that the function $\phi$ satisfies Kannappan’s condition $\phi(xyz) = \phi(xy)$ and $\varphi(xy) = \varphi(yx)$ for all $x, y, z \in G$.

Let $(G; +)$ be a topological abelian group and let $K$ be a compact subgroup of automorphisms of $G$ with the normalized Haar measure $\mu$. Assume that the topologies on $K$ and $G$ are related in such a way that the map $k \mapsto ky \in G, k \in K$ is continuous for each fixed $y \in G$, where $ky$ denotes the action of $k \in K$ on $y \in G$. We say that a continuous function $\varphi : G \to C$ is $K$-spherical if and only if there exists a non-zero continuous function $\varphi : G \to C$ such that

$$\int_K \phi(x + ky)d\mu(k) = \phi(x)\varphi(y)$$

for all $x, y \in G$. Equivalently, a non-zero continuous function $\varphi : G \to C$ is $K$-spherical if it satisfies the integral equation $\int_K \varphi(x+ky)d\mu(k) = \varphi(x)\varphi(y)$ for all $x, y \to G$. R. Badora[4] has studied the Hyers-Ulam stability of Wilson’s functional equation for spherical functions.

Classical examples of (1.1) are d’Alembert’s functional equation $\varphi(x + y) + \varphi(x - y) = 2\varphi(x)\varphi(y)$, where $K = \{Id, -Id\}$ and Cauchy’s equation $\varphi(x+y) = \varphi(x)\varphi(y)$ with $K = \{Id\}$. The generalization for (1.2) of Wilson’s functional equation (1.1) was considered discussed by W. Chojnacki [6], R.

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Badora [3, 4] and H. Stetkaer [17, 18]. For (1.2) with $K$ finite this problem was solved by W. Förorg-Rob and J. Schwaiger in [12], R. Badora in [3], and for d’Alembert’s functional equation by J. Baker in [5].

Several papers deal with Wilson’s functional equation, see e.g. the monograph [1] by Aczél and Dhombres for references and results. Aczél, Chung and Ng [2], where $K$ is a quadratically closed field of char $K \neq 2$, assuming that the function $g$ satisfies Kannappan’s condition, $\phi(xyz) = \phi(xzy)$ for all $x, y, z \in G$ and $\varphi(xy) = \varphi(yx)$. Penney and Rukhin [15] found square integrable solutions of a version of the equation (1.1). Sinopoulos [16] has determined the general solution of (1.1) where $G$ is a 2-divisible abelian group, $\varphi$ is a vector-valued function and $\phi$ is a matrix-valued function. Also, Wilson’s equation was investigated in the context of spherical functions on groups by Stetkaer [18]. In this paper we study the problem of the Hyers-Ulam stability of equation (1.1) for $K$ a $P_3$-Group, if the commutator subgroup $K_0$ of $K$, which is generated by all commutators $[x, y] := x^{-1}y^{-1}xy$, has order one or two.

2. Main results on stability

The main results on stability are contained in the following

**Theorem 1.** Let $\varphi, \phi : G \to K$ be continuous functions, where $G$ is a $P_3$-group and $K$ is a quadratically closed field with char $K \neq 2$; also $K$ is Abelian under multiplication. Assume that there exists a $c \geq 0$ such that

$$\|\varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x)\phi(y)\| \leq c, \quad x, y \in K$$

(2.1)

Then either

(i) $\varphi, \phi$ are bounded or
(ii) $\varphi$ is unbounded and

$$\phi(x) = \frac{1}{2} \lim_{n \to \infty} \varphi(u_n)^{-1}(\varphi(u_nx) + \varphi(u_nx^{-1})), \quad \phi(y) = \phi(y^{-1}), \quad \varphi(xy) + \varphi(xy^{-1}) = 2\varphi(x)\phi(y),$$

(2.2)

or

(iii) $\phi$ is unbounded,

$$\varphi(x) = \frac{1}{2} \lim_{n \to \infty} (\varphi(xu_n) + \varphi(xu_n^{-1}))\phi(u_n)^{-1},$$

(2.3)

and $\varphi, \phi$ satisfies (1.1).

**Corollary 1.** Let $\varphi : G \to K$ is continuous, if there exists a $c \geq 0$, then $\varphi$ is bounded or satisfied (1.1).
Consider the signed Wilson’s functional equation
\[ \varphi(xy\Lambda(xy)) + \varphi(xy^{-1}\Lambda(xy^{-1})) = 2\varphi(x)\phi(y), \] (2.5)
where \( \Lambda : G_0 \to T = \{a : |a| = e\} \) and \( G_0 \) is the commutator subgroup of group \( G \), which is generated by all commutators \([x, y] := x^{-1}y^{-1}xy\), has order one or two.

If we set \( \Lambda \equiv e \) with \( e \) is unit of \( K \), we can obtain (1.1).

**Theorem 2.** Let \( \varphi, \phi, G \to K \) be continuous functions and \( \Lambda \) be \( G \)-even, where \( G \) is a \( P_3 \)-group and \( K \) is a quadratically closed field with \( \text{char}\ K \neq 2 \), also \( K \) is Abelian under multiplication. Assume that there exists a \( c \geq 0 \) such that
\[ \|\varphi(xy\Lambda(xy)) + \varphi(xy^{-1}\Lambda(xy^{-1}))\| - 2\varphi(x)\phi(y) \leq c, \quad x, y \in K \] (2.6)
Then either
(i) \( \varphi, \phi \) are bounded or
(ii) \( \varphi \) is unbounded and
\[ \varphi(x) = \frac{1}{2} \lim_{n \to \infty} \varphi(u_n)^{-1}(\varphi(u_n x\Lambda(u_n x)) + \varphi(u_n x^{-1}\Lambda(u_n x^{-1}))), \] (2.7)
satisfies
\[ \varphi(x) = \phi(x^{-1}), \quad \phi(xy\Lambda(xy)) + \phi(xy^{-1}\Lambda(xy^{-1})) = 2\phi(x)\phi(y), \] (2.8)
or
(iii) \( \phi \) is unbounded,
\[ \varphi(x) = \frac{1}{2} \lim_{n \to \infty} (\varphi(xu_n\Lambda(xu_n)) + \varphi(xu_n^{-1}\Lambda(xu_n^{-1}))\phi(u_n)^{-1}, \] (2.9)
and \( \varphi, \phi \) satisfies (2.6).

3. Proofs of theorems

**Proof of Theorem 1.** Let
\[ f(x, y) = \varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x)\phi(y), \quad x, y \in G, \]
then we obtain
\[ \|f(x, y)\| \leq c, \quad x, y \in G. \] (3.10)
Furthermore, we get identities
\[ f(x, y) - f(x, y^{-1}) = \varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x)\phi(y) \]
\[ - [\varphi(xy^{-1}) + \varphi(xy) - 2\varphi(x)\phi(y)] \]
\[ = 2\varphi(x)(\phi(y^{-1}) - \phi(y)). \] (3.11)
If \( \varphi = 0 \), \( \phi \) is solution of (1.1).
If $\varphi$ is unbounded, from (3.11) we get $\phi(y) = \phi(y^{-1})$, i.e., $\phi$ is even. We now prove (ii) and (iii). Assuming (ii), there exists a sequence $\{u_n\}, m \in N$ in $K$ such that
\[ \varphi(u_n) \neq 0, \quad \text{and} \quad \lim_{n \to \infty} \|\varphi(u_n)\| = +\infty. \] (3.12)

Let $x = u_n$, $y = x$ in (2.1), we get
\[ \|\varphi(u_n x) + \varphi(u_n x^{-1}) - 2\varphi(u_n)\phi(x)\| \leq c, \quad x, y \in K. \] (3.13)
Then we obtain
\[ \|\varphi(u_n)^{-1}(\varphi(u_n x) + \varphi(u_n x^{-1})) - 2\phi(x)\| \leq \frac{c}{\|\varphi(u_n)\|}, \quad x, y \in K. \]
Consequently
\[ \lim_{n \to \infty} \varphi(u_n)^{-1}(\varphi(u_n x) + \varphi(u_n x^{-1})) = 2\phi(x). \] (3.14)
Now for each $x, y, z \in K$ and $n \in N$, by setting $x = u_n$, $y = xz$ in (2.1) we obtain
\[ \|\varphi(u_n xz) + \varphi(u_n(xz)^{-1}) - 2\varphi(u_n)\phi(xz)\| \leq c, \quad x, y \in K. \] (3.15)
then
\[ \lim_{n \to \infty} \varphi(u_n)^{-1}(\varphi(u_n xz) + \varphi(u_n(xz)^{-1})) = 2\phi(xz). \] (3.16)
By the arbitrariness of $z$, (3.14) converges to a unique function $\phi$ which satisfies (2.3). In fact,
\[
\begin{align*}
\|\varphi^{-1}(u_n)(\varphi(u_n xy) + \varphi(u_n(xy)^{-1})) + \varphi^{-1}(u_n)(\varphi(u_n y^{-1})
+ \varphi(u_n(xy^{-1})^{-1})) - 2\varphi^{-1}(u_n)(\varphi(u_n x) + \varphi(u_n x^{-1}))\phi(y))\|
\leq (\varphi^{-1}(u_n)(\varphi(u_n xy) + \varphi(u_n xy^{-1}) - 2\varphi(u_n x)\phi(y)))
+ ((\varphi(u_n(yx)^{-1}) + \varphi(u_n(xy^{-1})^{-1}) - 2\varphi(u_n x^{-1})\phi(y)))\|
\leq 2c\|\varphi^{-1}(u_n)\|,
\end{align*}
\] (3.17)
here we have used Kannappan’s condition on $\varphi$ to get (3.17). Then taking limits in (3.17) we get that $\phi$ satisfies (2.3). Hence (ii) is proved.

If $\phi$ is unbounded, there exists a sequence $\{u_n\}, m \in N$ in $K$ such that
\[ \phi(u_n) \neq 0, \quad u_n \neq 0, \quad \text{and} \quad \lim_{n \to \infty} \|\phi(u_n)\| = +\infty. \] (3.18)
By setting $y = u_n$ in (2.1) we obtain
\[ \|\varphi(xu_n) + \varphi(xu_n^{-1}) - 2\varphi(x)\phi(u_n)\| \leq c, \quad x, y \in K. \] (3.19)
Then we obtain
\[ \|\varphi(xu_n) + \varphi(xu_n^{-1})\phi(u_n)^{-1} - 2\phi(x)\| \leq \frac{c}{\|\phi(u_n)\|}, \quad x, y \in K. \]
Consequently
\[ \lim_{n \to \infty} (\varphi(xu_n) + \varphi(xu_n^{-1}))\varphi(u_n)^{-1} = 2\varphi(x). \] (3.20)

Now for each \( x, y, z \in K \) and \( n \in N \), by setting \( x = xz, y = u_n, \) in (2.1) we obtain
\[ \|\varphi(xzu_n) + \varphi((xz)u_n^{-1}) - 2\varphi(xz)\varphi(u_n)\| \leq c, \quad x, y \in K. \] (3.21)

Hence
\[ \lim_{n \to \infty} (\varphi(xzu_n) + \varphi((xz)u_n^{-1}))\varphi(u_n)^{-1} = 2\varphi(xz). \] (3.22)

By the arbitrariness of \( z \), (3.20) converges to a unique function \( \varphi \) which satisfies (1.1). In fact,
\[
\|f(u_nxy) + f(u_n(xy)^{-1})\varphi^{-1}(u_n) + (f(u_nxy^{-1}) + f(u_n(xy)^{-1}))\varphi^{-1}(u_n)
- 2\varphi(y)(\varphi(u_nx) + \varphi(u_nx^{-1}))\varphi^{-1}(u_n)\|
\leq \||\varphi(u_nxy) + \varphi(u_nxy^{-1})\varphi^{-1}(u_n) - 2\varphi(y)\varphi(u_nx)\varphi^{-1}(u_n)\|
+ \||\varphi(u_n(xy)^{-1}) + \varphi(u_n(xy)^{-1})\varphi^{-1}(u_n) - 2\varphi(y)\varphi(u_nx^{-1})\varphi^{-1}(u_n)\|
\leq 2c\|\varphi^{-1}(u_n)\|,
\] (3.23)

here we have used Kannappan’s condition on \( \varphi \) to get (3.23). Then taking limits in (3.23) we get that \( \phi \) satisfies (2.4), and \( \varphi, \phi \) satisfy (1.1). Then (iii) is proved. Then case (i) is also proved. \( \square \)

**Remark 3.1.** If \( \varphi \) is bounded, then in (iii) \( \varphi = 0 \), moreover, \( \varphi, \phi \) satisfy (2.1).

**Proof of Theorem 2.** Let
\[ f(x, y) = \varphi(xy\Lambda(xy)) + \varphi(xy^{-1}\Lambda(xy^{-1})) - 2\varphi(x)\phi(y), \quad x, y \in G, \]
then we obtain
\[ \|f(x, y)\| \leq c, \quad x, y \in G. \] (3.24)

Furthermore, we get identities
\[
f(x, y) - f(x, y^{-1}) = \varphi(xy\Lambda(xy)) + \varphi(xy^{-1}\Lambda(xy^{-1})) - 2\varphi(x)\phi(y)
- [\varphi(xy^{-1}\Lambda(xy^{-1})) + \varphi(xy\Lambda(xy)) - 2\varphi(x)\phi(y)]
= 2\varphi(x)(\phi(y^{-1}) - \phi(y)). \] (3.25)

If \( \varphi = 0 \), \( \varphi \) is solution of (2.5). If \( \varphi \) is unbounded, from (3.25) we get
\( \phi(y) = \phi(y^{-1}) \), i.e., \( \varphi \) is even.

Also there exists a sequence \( \{u_n\}, m \in N \) in \( K \) such that
\[ \varphi(u_n) \neq 0, \quad \text{and} \quad \lim_{n \to \infty} \|\varphi(u_n)\| = +\infty. \] (3.26)
Let $x = u_n, y = x$ in (2.6), we get
\[
\|\varphi(u_n x\Lambda(u_n x)) + \varphi(u_n x^{-1}\Lambda(u_n x^{-1})) - 2\varphi(u_n)g(x)\| \leq c, \quad x, y \in K. \quad (3.27)
\]
Then we obtain
\[
\|\varphi(u_n)^{-1}(\varphi(u_n x\Lambda(u_n x)) + \varphi(u_n x^{-1}\Lambda(u_n x^{-1})) - 2\varphi(u_n)g(x))\| \leq \frac{c}{\|\varphi(u_n)\|}, \quad x, y \in K.
\]
Consequently
\[
\lim_{n \to \infty} \varphi(u_n)^{-1}(\varphi(u_n x\Lambda(u_n x)) + \varphi(u_n x^{-1}\Lambda(u_n x^{-1}))) = 2\varphi(x). \quad (3.28)
\]
Now for each $x, y, z \in K$ and $n \in N$, by setting $x = u_n, y = xz$ in (2.6) we obtain
\[
\|\varphi(u_n xz\Lambda(u_n xz)) + \varphi(u_n x^{-1}\Lambda(u_n x^{-1})) - 2\varphi(u_n)\phi(xz)\| \leq c, \quad x, y \in K, \quad (3.29)
\]
then
\[
\lim_{n \to \infty} \varphi(u_n)^{-1}(\varphi(u_n xz\Lambda(u_n xz)) + \varphi(u_n x^{-1}\Lambda(u_n x^{-1}))) = 2\varphi(xz). \quad (3.30)
\]
By the arbitrariness of $z$, (3.28) converges to a unique function $\phi$ which satisfies (2.6). In fact,
\[
\|\varphi^{-1}(u_n)(\varphi(u_n x\Lambda(u_n x)) + \varphi(u_n x^{-1}\Lambda(u_n x^{-1}))) + \varphi^{-1}(u_n)(\varphi(u_n x y^{-1}\Lambda(u_n x y^{-1})) + \varphi(u_n x^{-1}\Lambda(u_n x^{-1}))) - 2\varphi^{-1}(u_n)(\varphi(u_n x + \varphi(u_n x^{-1})\phi(y))\|
\leq \|\varphi^{-1}(u_n)(\varphi(u_n x\Lambda(u_n x)) + \varphi(u_n x^{-1}\Lambda(u_n x^{-1}))) - 2\varphi(u_n x)\phi(y))\|
\leq \|\varphi^{-1}(u_n)(\varphi(u_n x y^{-1}\Lambda(u_n x y^{-1})) + \varphi(u_n x^{-1}\Lambda(u_n x^{-1}))) - 2\varphi(u_n x^{-1})\phi(y))\| \leq 2c\|\varphi^{-1}(u_n)\|. \quad (3.31)
\]
where we have used Kannappan’s condition on $\varphi$ and (2.6) to get (3.31). Then by taking limits in (3.31), we get that $\phi$ satisfies (2.8). Then (ii) is proved.

If $\phi$ is unbounded, there exists a sequence $\{u_n\}, m \in N$ in $K$ such that
\[
\phi(u_n) \neq 0, \quad u_n \neq 0, \quad \text{and} \quad \lim_{n \to \infty} \|\phi(u_n)\| = +\infty. \quad (3.32)
\]
By setting $y = u_n$ in (2.1) we obtain
\[
\|\varphi(xu_n\Lambda(xu_n)) + \varphi(xu_n^{-1}\Lambda(xu_n^{-1})) - 2\varphi(x)\phi(u_n)\| \leq c, \quad x, y \in K. \quad (3.33)
\]
Then we obtain
\[
\|\varphi(xu_n\Lambda(xu_n)) + \varphi(xu_n^{-1}\Lambda(xu_n^{-1}))\phi(u_n)^{-1} - 2\varphi(x)\| \leq \frac{c}{\|\phi(u_n)\|}, \quad x, y \in K.
\]
Consequently
\[
\lim_{n \to \infty} \left( \varphi(xu_n \Lambda(xu_n)) + \varphi(xy_n^{-1} \Lambda(xy_n^{-1})) \right) \phi(u_n)^{-1} = 2\varphi(x). \tag{3.34}
\]
Now for each \( x, y, z \in K \) and \( n \in N \), by setting \( x = xy, y = u, \) in (2.6) we obtain
\[
\| \varphi(xy_n \Lambda(xy_n)) + \varphi(xy_n^{-1} \Lambda(xy_n^{-1})) - 2\varphi(xy)\phi(u_n) \| \leq c, \quad x, y \in K. \tag{3.35}
\]
then
\[
\lim_{n \to \infty} \left( \varphi(xy_n \Lambda(xy_n)) + \varphi(xy_n^{-1} \Lambda(xy_n^{-1})) \right) \phi(u_n)^{-1} = 2\varphi(xy). \tag{3.36}
\]
By the arbitrariness of \( y \), (3.34) converges to a unique function \( \varphi \) which satisfies (2.6). In fact,
\[
\| (\varphi(u_nxy \Lambda(u_nxy)) + \varphi(u_nxy^{-1} \Lambda(u_nxy^{-1})) \| \phi(u_n)^{-1}
\]
\[
+ \| (\varphi(u_nxy^{-1} \Lambda(u_nxy^{-1})) + \varphi(u_nyx^{-1}) \Lambda(\varphi(u_nyx^{-1})) \| \phi(u_n)^{-1}
\]
\[
- 2\varphi(y)\phi(u_nx) + \varphi(u_nx^{-1})\| \phi(u_n)^{-1}
\]
\[
\leq \| (\varphi(u_nxy \Lambda(u_nxy)) + \varphi(u_nxy^{-1} \Lambda(u_nxy^{-1})) - 2\varphi(y)\phi(u_nx))\phi(u_n)^{-1}\|
\]
\[
+ \| (\varphi(u_nxy^{-1} \Lambda(u_nxy^{-1})) + \varphi(u_nyx^{-1}) \Lambda(\varphi(u_nyx^{-1})) \| \phi(u_n)^{-1}
\]
\[
- 2\varphi(y)\phi(u_nx^{-1})\| \phi(u_n)^{-1}\| \leq 2c\| \varphi^{-1}(u_n) \|, \tag{3.37}
\]
where we have used Kannappan’s condition on \( \varphi, \Lambda \) and (2.6) to get (3.37). Then taking limits in (3.37), we get that \( \varphi \) satisfies (2.5). Therefore (iii) is proved. Then the case (i) is also proved. \( \square \)

4. Example

Example. Let \( C \) the field of complex numbers with the complex unit \( i = \sqrt{-1} \), and \( G \) be the quaternion group \( G = \{ \pm 1, \pm i, \pm j, \pm k \} \). The center of \( G \) is \( G_0 = \{ \pm 1 \} \) and \( G \) is a \( P_3 \)-group. Take \( \Lambda = Id \) or \(-Id\), \( \varphi, \phi : G \to C, \varphi \neq 0 \). If \( \phi \) is unbounded and \( \varphi, \phi \) satisfy (2.3) or (2.6), then \( \varphi \) as defined by (2.4) or (2.9) and \( \phi \) are solutions of (1.1) or (2.5) respectively.

References


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