THE DIAMETER OF A ZERO-DIVISOR GRAPH FOR FINE DIRECT PRODUCT OF COMMUTATIVE RINGS

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Abstract. This paper establishes a set of theorems that describe the diameter of a zero-divisor graph for a finite direct product $R_1 \times R_2 \times \cdots \times R_n$ with respect to the diameters of the zero-divisor graphs of $R_1, R_2, \ldots, R_{n-1}$ and $R_n (n > 2)$.

1. Introduction

All rings in this paper are commutative and not necessary with 1. The concept of zero divisor graph of a commutative ring $R$ was introduced by Beck in [2]. He let all elements of the ring be vertices of the graph and was interested mainly in coloring. In [1], Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are the non-zero zero-divisors. Among other things, they proved that $\Gamma(R)$ is always connected and its diameter is always less than or equal to 3 [1, Theorem 2.3]. The zero-divisor graph helps us to study the algebraic properties of rings using graph theoretical tools (see, for example, [1], [3], [4]). In [5], J. Warfel describes the diameter of a zero-divisor graph for a direct product $R_1 \times R_2$ with respect to the diameters of the zero-divisor graphs of $R_1$ and $R_2$. The main goal in this paper is to generalize some of the results in the paper listed as [5], from $R_1 \times R_2$ to $R_1 \times R_2 \times \cdots \times R_n (n > 2)$ (see section 2).

For the sake of completeness, we state some definitions and notations used throughout. Let $R$ be a commutative ring. We used $Z(R)$ to denote the set of zero-divisors of $R$; we use $Z^*(R)$ to denote the set of non-zero zero-divisors of $R$. By the zero-divisor graph of $R$, denoted $\Gamma(R)$, we mean the graph whose vertices are the non-zero zero-divisors of $R$, and for distinct $x, y \in Z^*(R)$, there is an edge connecting $x$ and $y$ if and only if $xy = 0$. A graph is said to be connected if there exists a path between any two distinct vertices. For two distinct vertices $a$ and $b$ in the graph $\Gamma(R)$, the
Let \( \text{diam}(\Gamma((i)) \) denote the diameter of the graph of \( Z^*(R) \). The diameter is zero if the graph consists of a single vertex and a connected graph with more than one vertex has diameter 1 if and only if it is complete; i.e. each pair of distinct vertices forms an edge. We tacitly assume that \( R \) has at least 2 non-zero zero-divisors. Also, though it be an abuse of notation, let \( 0 = (0, 0, \cdots, 0) \).

2. Finite direct product

In this section, we will investigate the relation between the diameter of a zero-divisor graph of a finite direct product \( R_1 \times R_2 \times \cdots \times R_n \) with the diameters of the zero-divisor graphs of \( R_1, R_2, \ldots, R_{n-1} \) and \( R_n \). Our starting point is the following lemma:

**Lemma 2.1.** Let \( R \) be commutative ring with \( \text{diam}(\Gamma(R)) = 1 \) and \( R = Z(R) \). Then \( xy = 0 \) for all \( x, y \in Z(R) \). In particular, \( x^2 = 0 \) for every nilpotent element of \( R \).

**Proof.** Suppose not. Then there are elements \( a, b \in Z(R) \) such that \( ab \neq 0 \), so by [1, Theorem 2.8], \( R \cong Z_2 \times Z_2 \); hence \( R \neq Z(R) \) which is a contradiction, as required. \( \square \)

**Theorem 2.2.** Let \( R_1, R_2, \ldots, R_{n-1} \) and \( R_n \) be commutative rings such that \( \text{diam}(\Gamma(R_1)) = \cdots = \text{diam}(\Gamma(R_n)) = 1 \), and let \( R = R_1 \times R_2 \times \cdots \times R_n \) \((n > 2)\). Then the following hold:

(i) \( \text{diam}(\Gamma(R)) = 1 \) if and only if \( R_i = Z(R_i) \) for every \( i \in \{1, \ldots, n\} \).

(ii) \( \text{diam}(\Gamma(R)) = 2 \) if and only if \( R_i = Z(R_i) \) and \( R_j \neq Z(R_j) \) for some \( i, j \in \{1, 2, \ldots, n\} \).

(iii) \( \text{diam}(\Gamma(R)) = 3 \) if and only if \( R_i \neq Z(R_i) \) for every \( i \in \{1, 2, \ldots, n\} \).

**Proof.** (i) Assume that \( R_i = Z(R_i) \) for every \( i = 1, 2, \ldots, n \) and let \( a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \) be elements of \( Z^*(R) \). By Lemma 2.1, \( a_i, b_i = 0 \) for all \( i \), so \( ab = 0 \); hence \( \text{diam}(\Gamma(R)) = 1 \). Conversely, assume that \( R_j \neq Z(R_j) \) for some \( j \in \{1, 2, \ldots, n\} \). Then, for some \( x_j, y_j \in R_j, x_jy_j \neq 0 \). Set \( x = (0, \ldots, x_j, 0, \ldots, 0), y = (0, \ldots, y_j, 0, \ldots, 0) \), and let \( 0 \neq a_i \in R_i \) where \( i \neq j \). Since \( x(0, \ldots, a_i, 0, \ldots, 0) = 0, y(0, \ldots, a_i, 0, \ldots, 0) = 0 \) and \( xy \neq 0 \), we must have \( \text{diam}(\Gamma(R)) > 1 \) which is a contradiction.

(ii) If \( R_i = Z(R_i) \) and \( R_j \neq Z(R_j) \) for some \( i, j \in \{1, 2, \ldots, n\} \), then by (i), the fact that \( R_j \neq Z(R_j) \) implies that \( \text{diam}(\Gamma(R)) > 1 \). Then there exist \( r = (r_1, \ldots, r_n) \in Z^*(R) \) and \( s = (s_1, \ldots, s_n) \in Z^*(R) \) such that \( d(r, s) \neq 1, so rs \neq 0 \). Since \( R_i = Z(R_i) \), there must exist \( t_i \in R_i \) such that
Let \( d_{\text{diam}}(\Gamma) = 0 \), which is a contradiction. Thus \( c \neq a \).

Proof. There must exist \( x_i \in R_i - Z(R_i) \) for every \( i \in \{1, 2, \ldots, n\} \). Let for each \( i \), \( z_i \in Z^*(R_i) \). So there is an element \( z_i' \) of \( Z^*(R_i) \) such that \( z_i z_i' = 0 \) for all \( i \). If \( a = (z_1, x_2, \ldots, x_n) \) and \( b = (x_1, z_2, x_3, \ldots, x_n) \), then \( a(z_1', 0, \ldots, 0) = 0 \) and \( b(0, z_2', 0, \ldots, 0) = 0 \), so \( ab \neq 0 \), the distance between the vertices is greater than one. Since \( d_{\text{diam}}(\Gamma) = 1 \), there must be some \( c = (c_1, \ldots, c_n) \in Z^*(R) \) such that \( ac = bc = 0 \). Then \( c = 0 \), which is not an element of \( Z^*(R) \). But this is a contradiction. Thus \( R_i = Z(R_i) \) and \( R_j \neq Z(R_j) \) for some \( i, j \in \{1, 2, \ldots, n\} \).

(iii) This follows from (i) and (ii).

We will need the following lemma from [5, Lemma 3.1].

**Lemma 2.3.** Let \( R \) be a commutative ring such that \( d_{\text{diam}}(\Gamma) = 2 \) and \( R = Z(R) \). Then for all \( x, y \in R \), there exists an element \( z \) of \( Z^*(R) \) such that \( xz = yz = 0 \).

**Theorem 2.4.** Let \( R_1, R_2, \ldots, R_{n-1} \) and \( R_n \) be commutative rings such that \( d_{\text{diam}}(\Gamma(R_1)) = \cdots = d_{\text{diam}}(\Gamma(R_n)) = 2 \), and let \( R = R_1 \times R_2 \times \cdots \times R_n \) \( (n > 2) \). Then the following hold:

(i) \( d_{\text{diam}}(\Gamma) \neq 1 \).

(ii) \( d_{\text{diam}}(\Gamma) = 2 \) if and only if \( R_i = Z(R_i) \) for some \( i \in \{1, 2, \ldots, n\} \).

(iii) \( d_{\text{diam}}(\Gamma) = 3 \) if and only if \( R_i \neq Z(R_i) \) for every \( i \in \{1, 2, \ldots, n\} \).

**Proof.** (i) Is clear.

(ii) Let \( R_i = Z(R_i) \) for some \( i \in \{1, 2, \ldots, n\} \). By (i), there are elements \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) of \( Z^*(R) \) such that \( x \neq y \) and \( xy \neq 0 \). Since \( x_i, y_i \in R_i \), Lemma 2.3 gives \( x_i z_i = 0 = y_i z_i \) for some non-zero element \( z_i \) of \( Z(R_i) \). Let \( z = (0, \ldots, z_i, 0, \ldots, 0) \). Since \( xz = 0 = yz \), we must have \( x = z - y \) is a path; hence a path of length two can be found between any two vertices of \( \Gamma(R) \) by way of \( z \). So, \( d_{\text{diam}}(\Gamma) = 2 \). Conversely, assume that \( d_{\text{diam}}(\Gamma) = 2 \) and let \( R_i \neq Z(R_i) \) for each \( i \in \{1, 2, \ldots, n\} \). Let for each \( i \), \( e_i \in Z^*(R_i) \) and \( m_i \in R_i - Z(R_i) \). So there is an element \( e_i' \) of \( Z^*(R_i) \) such that \( e_i e_i' = 0 \) for all \( i \). If \( a = (e_1, m_2, \ldots, m_n) \) and \( b = (m_1, e_2, m_3, \ldots, m_n) \), then \( a(e_1', 0, \ldots, 0) = 0 \) and \( b(0, e_2', 0, \ldots, 0) = 0 \), so \( ab \neq 0 \), the distance between the vertices is greater than one. Since \( d_{\text{diam}}(\Gamma) = 2 \), there must be some \( c = (c_1, \ldots, c_n) \in Z^*(R) \) such that \( ac = 0 = bc \). Then \( c = 0 \), which is a contradiction. Thus \( R_i \neq Z(R_i) \) for some \( i \in \{1, 2, \ldots, n\} \).

(iii) This follows from (i) and (ii).
Theorem 2.5. Let $R_1, R_2, \ldots, R_{n-1}$ and $R_n$ be commutative rings such that $\text{diam}(\Gamma(R_1)) = \cdots = \text{diam}(\Gamma(R_{n-1})) = 3$, and let $R = R_1 \times R_2 \times \cdots \times R_n$ ($n > 2$). Then $\text{diam}(\Gamma(R)) = 3$.

Proof. Since for each $i \in \{1, 2, \ldots, n\}$, $\text{diam}(\Gamma(R_i)) = 3$, there exist $x_i, y_i \in Z^*(R_i)$ with $x_i \neq y_i, x_iy_i \neq 0$ such that there is no $z_i \in Z^*(R_i)$ with $x_iz_i = 0 = y_iz_i$. Consider $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. For each $i \in \{1, 2, \ldots, n\}$, there are elements $x'_i, y'_i \in Z^*(R_i)$ such that $x_ix'_i = 0$ and $y iy'_i = 0$, so $x, y \in Z^*(R)$. As $xy \neq 0$, we must have $\text{diam}(\Gamma(R)) \neq 1$. If $\text{diam}(\Gamma(R)) = 2$, then $d(x, y) \neq 1$ implies there is an element $a = (a_1, \ldots, a_n) \in Z^*(R)$ with $xa = 0 = ya$; hence $a = 0$ by our assumption which is a contradiction, so $\text{diam}(\Gamma(R)) = 3$ must hold.

Theorem 2.6. Let $R_1, R_2, \ldots, R_{n-1}$ and $R_n$ be commutative rings such that $\text{diam}(\Gamma(R(i))) = 1$, $\text{diam}(\Gamma(R(j))) = 2$ for some $i, j \in \{1, 2, \ldots, n\}$ and there is no $k \in \{1, 2, \ldots, n\}$ with $\text{diam}(\Gamma(R_k)) = 3$, and let $R = R_1 \times R_2 \times \cdots \times R_n$ ($n > 2$). Then the following hold:

1. $\text{diam}(\Gamma(R)) \neq 1$.
2. $\text{diam}(\Gamma(R)) = 2$ if and only if $R_i = Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$.
3. $\text{diam}(\Gamma(R)) = 3$ if and only if $R_i \neq Z(R_i)$ for every $i \in \{1, 2, \ldots, n\}$.

Proof. (i) Is clear.

(ii) First, assume that $R_i = Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$; we show that $\text{diam}(\Gamma(R)) = 2$. By hypothesis, we divided the proof into two cases.

Case 1. $\text{diam}(\Gamma(R_i)) = 1$. It then follows from Lemma 2.2 that $xy = 0$ for all $x, y \in Z(R_i)$. By (i), there must exist $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in Z^*(R)$ with $xy \neq 0$. If $z_i \in Z^*(R_i)$, then $x(0, \ldots, z_i, \ldots, 0) = 0$, so $z = (0, \ldots, z_i, \ldots, 0)$ is an element of $Z^*(R)$. Clearly, $x - z - y$ is a path. Hence, a path of length two can be found between any two vertices of $\Gamma(R)$ by way of $z$. So, $\text{diam}(\Gamma(R)) = 2$.

Case 2. $\text{diam}(\Gamma(R_i)) = 2$. By (i), there must exist $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in Z^*(R)$ with $xy \neq 0$. By Lemma 2.3, there is an element $z_i$ of $Z^*(R_i)$ such that $x_iz_i = y_iz_i = 0$. Set $z = (0, \ldots, z_i, 0, \ldots, 0)$. Then $x - z - y$ is a path, and hence a path of length two can be found between any two vertices of $\Gamma(R)$ by way of $z$. So, $\text{diam}(\Gamma(R)) = 2$.

Next assume that $\text{diam}(\Gamma(R)) = 2$; we show that $R_i = Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$. Suppose that for each $i \in \{1, 2, \ldots, n\}$, $R_i \neq Z(R_i)$. Let for each $i$, $x_i \in Z^*(R_i)$ and $m_i \in R_i - Z(R_i)$. So there is an element $x'_i$ of $Z^*(R_i)$ such that $x_ix'_i = 0$ for all $i$. If $a = (x_1, m_2, \ldots, m_n)$ and $b = (m_1, x_2, m_3, \ldots, m_n)$, then $a(x'_1, 0, \ldots, 0) = 0$ and $b(0, x'_2, 0, \ldots, 0) = 0$, so $a, b \in Z^*(R)$. As $ab \neq 0$, the distance between the vertices is greater than one. Since $\text{diam}(\Gamma(R)) = 2$, there must be some $c = (c_1, \ldots, c_n) \in Z^*(R)$
such that $ac = 0 = bc$. Then $c = 0$, which is a contradiction. Thus $R_i \neq Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$.

(iii) This follows from (i) and (ii).

Theorem 2.7. Let $R_1, R_2, \ldots, R_{n-1}$ and $R_n$ be commutative rings such that $\text{diam}(\Gamma(R_i)) = 1$, $\text{diam}(\Gamma(R_j)) = 3$ for some $i, j \in \{1, 2, \ldots, n\}$ and there is no $k \in \{1, 2, \ldots, n\}$ with $\text{diam}(\Gamma(R_k)) = 2$, and let $R = R_1 \times R_2 \cdots \times R_n$ ($n > 2$). Then the following hold:

(i) $\text{diam}(\Gamma(R)) \neq 1$.

(ii) $\text{diam}(\Gamma(R)) = 2$ if and only if $\text{diam}(\Gamma(R_i)) = 1$ and $R_i = Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$.

(iii) $\text{diam}(\Gamma(R)) = 3$ if and only if there is no $i \in \{1, 2, \ldots, n\}$ with $\text{diam}(\Gamma(R_i)) = 1$ and $R_i = Z(R_i)$.

Proof. (i) Is clear.

(ii) Let $i$ be such that $\text{diam}(\Gamma(R_i)) = 1$ and $R_i = Z(R_i)$; we show that $\text{diam}(\Gamma(R)) = 2$. It follows from [1, Theorem 2.8] that $a_i b_i = 0$ for every $a_i, b_i \in Z(R_i)$. By (i), there must exist $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in Z^*(R)$ such that $xy \neq 0$. Assume that $a_i \in Z^*(R_i)$ and set $a = (0, 0, \ldots, a_i, 0, 0)$. Then $ax = 0 = ay$, so $a \in Z^*(R)$. Therefore, $x - a - y$ is a path, and hence a path of length two can be found between any two vertices of $\Gamma(R)$ by way of $a$. So $\text{diam}(\Gamma(R)) = 2$. Conversely, assume that $\text{diam}(\Gamma(R)) = 2$; we show that $\text{diam}(\Gamma(R_i)) = 1$ and $R_i = Z(R_i)$ for some $i$. Suppose not. Let $i_1, \ldots, i_k$ be such that $\text{diam}(\Gamma(R_{i_r})) = 1$ ($1 \leq r \leq k$), and let $j_1, \ldots, j_t$ be such that $\text{diam}(\Gamma(R_{j_s})) = 3$ ($1 \leq s \leq t$). Since for each $s$ ($1 \leq s \leq t$), $\text{diam}(\Gamma(R_{j_s})) = 3$, there exist $x_{j_s}, y_{j_s} \in Z^*(R_{j_s})$ with $x_{j_s} \neq y_{j_s}, x_{j_s}y_{j_s} = 0$ such that there is no $z_{j_s} \in Z^*(R_{j_s})$ with $x_{j_s}z_{j_s} = y_{j_s}z_{j_s} = 0$. Moreover, for each $s$ ($1 \leq s \leq t$), there must exist $x_{j_s}'_{j_s}, y_{j_s}'_{j_s} \in Z^*(R_{j_s})$ with $x_{j_s}'_{j_s}y_{j_s}'_{j_s} = 0$ and $y_{j_s}'_{j_s} = 0$. Now for each $r$ ($1 \leq r \leq k$), let $m_{i_r} \in R_{i_r} - Z(R_{i_r})$. Set $c = (m_{i_1}, \ldots, x_{j_1}, \ldots, x_{j_t}, \ldots)$ and $d = (m_{i_1}, \ldots, y_{j_1}, \ldots, y_{j_t}, \ldots)$. Then $c(0, \ldots, x_{j_1}'_{j_1}, 0, \ldots, 0) = 0$, so $c \in Z^*(R_i)$. Similarly, $d \in Z^*(R_i)$. As $cd \neq 0$ and $\text{diam}(\Gamma(R)) = 2$, there must be some $e = (e_1, \ldots, e_n) \in Z^*(R)$ such that $ce = de = 0$. Then $e = 0$, which is a contradiction. Thus $\text{diam}(\Gamma(R_i)) = 1$ and $R_i = Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$.

(iii) Since $\Gamma(R)$ is connected and $\text{diam}(\Gamma(R)) \leq 3$, we must have the diameter of $\Gamma(R)$ is either 2 or 3 by (i). If $\text{diam}(\Gamma(R)) = 2$, then by (ii), $\text{diam}(\Gamma(R_i)) = 1$ and $R_i = Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$ which is a contradiction. Thus $\text{diam}(\Gamma(R)) = 3$. The proof of the other implication is clear.

Theorem 2.8. Let $R_1, R_2, \ldots, R_{n-1}$ and $R_n$ be commutative rings such that $\text{diam}(\Gamma(R_i)) = 2$, $\text{diam}(\Gamma(R_j)) = 3$ for some $i, j \in \{1, 2, \ldots, n\}$ and there is
no \( k \in \{1, 2, \ldots, n\} \) with \( \text{diam}(\Gamma(R_k)) = 1 \), and let \( R = R_1 \times R_2 \cdots \times R_n \) \((n > 2)\). Then the following hold:

(i) \( \text{diam}(\Gamma(R)) \neq 1 \).

(ii) \( \text{diam}(\Gamma(R)) = 2 \) if and only if \( \text{diam}(\Gamma(R_i)) = 2 \) and \( R_i = Z(R_i) \) for some \( i \in \{1, 2, \ldots, n\} \).

(iii) \( \text{diam}(\Gamma(R)) = 3 \) if and only if there is no \( i \in \{1, 2, \ldots, n\} \) with \( \text{diam}(\Gamma(R_i)) \leq 2 \) and \( R_i = Z(R_i) \).

Proof. (i) Is clear.

(ii) Let \( i \) be such that \( \text{diam}(\Gamma(R_i)) = 2 \) and \( R_i = Z(R_i) \); we show that \( \text{diam}(\Gamma(R)) = 2 \). Then by Lemma 2.3, \( a_i b_i = 0 \) for every \( a_i, b_i \in Z(R_i) = R_i \). By (i), there must exist \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in Z^*(R) \) such that \( xy \neq 0 \). Assume that \( a_i \in Z^*(R_i) \) and set \( a = (0, 0, \ldots, a_i, 0, \ldots, 0) \). Then \( ax = 0 = ay \), so \( a \in Z^*(R) \). Therefore \( x - a - y \) is a path, and hence a path of length two can be found between any two vertices of \( \Gamma(R) \) by way of \( a \). So, \( \text{diam}(\Gamma(R)) = 2 \). Conversely, assume that \( \text{diam}(\Gamma(R)) = 2 \); we show that \( \text{diam}(\Gamma(R_i)) = 2 \) and \( R_i = Z(R_i) \) for some \( i \). Suppose that for each \( i \) \((1 \leq i \leq n)\), if \( \text{diam}(\Gamma(R_i)) = 2 \), then \( R_i \neq Z(R_i) \). Let \( i_1, \ldots, i_k \) be such that \( \text{diam}(\Gamma(R_{i_1})) = 2 \) \((1 \leq i \leq k)\), and let \( j_1, \ldots, j_l \) be such that \( \text{diam}(\Gamma(R_{j_s}))(1 \leq s \leq t) \). By assumption, for each \( r \) \((1 \leq r \leq k)\), \( R_{i_r} \neq Z(R_{i_r}) \). For each \( r \) \((1 \leq r \leq k)\), let \( m_{i_r} \in R_{i_r} - Z(R_{i_r}) \). Since for each \( s \) \((1 \leq s \leq t)\), \( \text{diam}(\Gamma(R_{j_s})) = 2 \), there exist \( x_{j_s}, y_{j_s} \in Z^*(R_{j_s}) \) with \( x_{j_s} \neq y_{j_s}, x_{j_s}y_{j_s} = 0 \) such that there is no \( z_{j_s} \in Z^*(R_{j_s}) \) with \( x_{j_s}z_{j_s} = 0 = y_{j_s}z_{j_s} \). Moreover, for each \( s \) \((1 \leq s \leq t)\), there must exist \( x_{j_s}, y_{j_s} \in Z^*(R_{j_s}) \) with \( x_{j_s}x_{j_s}' = 0 \) and \( y_{j_s}y_{j_s}' = 0 \). Set \( c = (m_{i_1}, \ldots, x_{j_1}, \ldots, x_{j_l}, \ldots) \) and \( cd = (m_{i_1}, \ldots, y_{j_1}, \ldots, y_{j_l}, \ldots) \). Then \( c(0, \ldots, x_{j_1}, 0, \ldots, 0) = 0 \), so \( c \in Z^*(R) \). Similarly, \( d \in Z^*(R) \). As \( cd \neq 0 \) and \( \text{diam}(\Gamma(R)) = 2 \), there must be some \( e = (e_1, \ldots, e_n) \in Z^*(R) \) such that \( ce = 0 = de \). Then \( e = 0 \), which is a contradiction. Thus \( \text{diam}(\Gamma(R_i)) = 2 \) and \( R_i = Z(R_i) \) for some \( i \in \{1, 2, \ldots, n\} \).

(iii) This follow from (i) and (ii). \(\square\)

Theorem 2.9. Let \( R_1, R_2, \ldots, R_{n-1} \) and \( R_n \) be commutative rings such that \( \text{diam}(\Gamma(R_i)) = 1 \), \( \text{diam}(\Gamma(R_j)) = 2 \) and \( \text{diam}(\Gamma(R_k)) = 3 \) for some elements \( i, j \) and \( k \) of the set \( \{1, 2, \ldots, n\} \), and let \( R = R_1 \times R_2 \cdots \times R_n \) \((n > 2)\). Then the following hold:

(i) \( \text{diam}(\Gamma(R)) \neq 1 \).

(ii) \( \text{diam}(\Gamma(R)) = 2 \) if and only if \( \text{diam}(\Gamma(R_i)) \leq 2 \) and \( R_i = Z(R_i) \) for some \( i \in \{1, 2, \ldots, n\} \).

(iii) \( \text{diam}(\Gamma(R)) = 3 \) if and only if there is no \( i \in \{1, 2, \ldots, n\} \) with \( \text{diam}(\Gamma(R_i)) \leq 2 \) and \( R_i = Z(R_i) \).

Proof. (i) Is clear.
(ii) Let $\text{diam}(\Gamma(R_i)) \leq 2$ and $R_i = Z(R_i)$ for some $i \in \{1, 2, \ldots, n\}$; we show that $\text{diam}(\Gamma(R)) = 2$. We divided the proof into two cases.

**Case 1.** $\text{diam}(\Gamma(R_i)) = 1$ and $R_i = Z(R_i)$ for some $i$. By a similar argument as in Theorem 2.7 (ii), we get $\text{diam}(\Gamma(R)) = 2$.

**Case 2.** $\text{diam}(\Gamma(R_i)) = 2$ and $R_i = Z(R_i)$ for some $i$. By a similar argument as in Theorem 2.8 (ii), we get $\text{diam}(\Gamma(R)) = 2$. Conversely, suppose that $\text{diam}(\Gamma(R)) = 2$. It is easy to see from Theorem 2.8 (ii) that $\text{diam}(\Gamma(R_i)) \leq 2$ and $R_i = Z(R_i)$ for some $i$.

(iii) This follow from (i) and (ii).

**Corollary 2.10.** Let $R_1, R_2, \ldots, R_{n-1}$ and $R_n$ be commutative rings with identity, and let $R = R_1 \times \cdots \times R_n$ ($n > 2$). Then $\text{diam}(\Gamma(R)) = 3$.

**Proof.** For each $i \in \{1, 2, \ldots, n\}$, $R_i \neq Z(R_i)$ since $1_{R_i} \notin Z(R_i)$. Now the assertion follows from Theorem 2.2, Theorem 2.4 and Theorem 2.6 (for an alternative proof see [3, 2.6 (4)]).

**Example 1.** (i) Assume that $R$ is a commutative ring (not necessary with 1) and let $S = \text{Mat}(R)$ be the set of all $2 \times 2$ matrices of the form

\[
A = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}
\]

where $a \in R$. It is easy to see that if $A, B$ are non-zero elements of $S$, then $AB = 0$; hence $Z(S) = S$ and $\text{diam}(\Gamma(S)) = 1$.

(ii) Let $Z_{25}$ denote the ring of integers modulo 25. Then $Z^*(Z_{25}) = \{5, 10, 15, 20\}$, $Z_{25} \neq Z(Z_{25})$ and $\text{diam}(\Gamma(Z_{25})) = 1$. Clearly, $Z(Z_2 \times Z_4) \neq Z_2 \times Z_4$ and $\text{diam}(\Gamma(Z_2 \times Z_4)) = 3$.

(iii) If $R_1 = R_2 = \cdots = R_n = S$ and $R = R_1 \times \cdots \times R_n$, then $\text{diam}(\Gamma(R)) = 1$ by Theorem 2.2 (i).

(iv) If $R_1 = Z_{25}$, $R_2 = \cdots = R_n = S$ and $R = R_1 \times \cdots \times R_n$, then $\text{diam}(\Gamma(R)) = 2$ by Theorem 2.2 (ii).

(v) If $R_1 = Z_{25}$, $R_2 = \cdots = R_n$ and $R = R_1 \times \cdots \times R_n$, then $\text{diam}(\Gamma(R)) = 3$ by Theorem 2.2 (iii).

(vi) If $R_1 = Z_2 \times Z_4$, $R_2 = \cdots = R_n$ and $R = R_1 \times \cdots \times R_n$, then $\text{diam}(\Gamma(R)) = 3$ by Theorem 2.5 (or Corollary 2.10).

**References**


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