DERIVATIVE-FREE CHARACTERIZATIONS OF $Q_K$ SPACES II

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Abstract. The $Q_K$ spaces on the open unit disk are characterized by some oscillations in the Bergman metric without the use of derivatives. Our results are new even in the case of $Q_p$ spaces.

1. Introduction

Let $\mathbb{D} = \{ z : |z| < 1 \}$ be the unit disk of complex plane $\mathbb{C}$. A particular class of M"obius invariant function spaces, the so-called $Q_p$ spaces, has attracted a lot of attention in recent years. Denote by $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$. For $a \in \mathbb{D}$, $g(z, a) = \log \frac{1}{|\phi_a(z)|}$ is the Green function in $\mathbb{D}$, where $\phi_a(z) = (a - z)/(1 - \bar{a}z)$ is the M"obius map of $\mathbb{D}$. For $0 \leq p < \infty$, the space $Q_p$ consists of all functions $f \in H(\mathbb{D})$ such that

$$
\|f\|_{Q_p}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g^p(z, a) dA(z) < \infty,
$$

where $dA$ is an area measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$. We denote $d\tau(z) = \frac{dA}{(1 - |z|^2)^2}$ the M"obius invariant measure. We know that the Green function $g(a, z)$ in (1) can be replaced by the expression $1 - |\phi_a(z)|^2$ (cf. [1]). It is well known that $Q_1 = BMOA$ and $Q_0$ is the classical Dirichlet space $D$ with

$$
\|f\|_D^2 = \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty.
$$

For $1 < p < \infty$, $Q_p = \mathcal{B}$. Here $\mathcal{B}$ is the Bloch space defined as follows.

$$
\mathcal{B} = \{ f : f \in H(\mathbb{D}), \|f\|_\mathcal{B} = \sup_{a \in \mathbb{D}} |f'(z)|(1 - |z|^2) < \infty \}.
$$

For more about $Q_p$ spaces see [1], [2] and [7].

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For any nonnegative, nondecreasing and Lebesgue measurable function \( K \) on \((0, 1]\), we say that \( f \) belongs to the space \( Q_K \) if

\[
\|f\|_{Q_K}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2) \, dA(z) < \infty.
\]  

The space \( Q_{K,0} \) consists of analytic functions \( f \) on \( \mathbb{D} \) for which

\[
\lim_{|a| \to 1} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2) dA(z) = 0.
\]

Modulo constants, \( Q_K \) is a Banach space under the norm \( |f(0)| + \|f\|_{Q_K} \) and \( Q_K \) is Möbius invariant; that is, \( \|f \circ \varphi_a\|_{Q_K} = \|f\|_{Q_K} \) whenever \( f \in Q_K \) and \( a \in \mathbb{D} \). It is easy to see that \( Q_{K,0} \) is a closed subspace in \( Q_K \). For \( 0 < p < \infty \), \( K(t) = t^p \) gives the space \( Q_p \). \( K(t) = 1 \) gives the Dirichlet space \( \mathcal{D} \). More results on \( Q_K \) spaces can be found in [3], [4] and [5].

Recently, the second author and Zhu [6] investigated some free - derivative characterizations of the spaces \( Q_k \) and \( Q_{K,0} \). In this paper, we continue to give some free - derivative characterizations of the spaces \( Q_k \) and \( Q_{K,0} \).

2. Preliminaries

If the function \( K \) is only defined on \((0, 1]\), then we extend it to \((0, \infty)\) by setting \( K(t) = K(1) \) for \( t > 1 \). We define an auxiliary function (see [4] and [6]) as follows:

\[
\varphi_K(s) = \sup_{0 < t \leq 1} \frac{K(st)}{K(t)}, 0 < s < \infty.
\]  

We assume that \( K \) is continuous and nondecreasing on \((0, 1]\). This ensures that the above function is continuous and nondecreasing on \((0, \infty)\).

The following Lemma is very useful in the proof of the main theorem in [6].

**Lemma 2.1.** Let \( K \) be any nonnegative and Lebesgue measurable function on \((0, \infty)\) and \( f(z) = K(1 - |z|^2) \). If

\[
\int_0^\infty \frac{\varphi_K(x)}{(1 + x)^3} \, dx < \infty,
\]

then there exists a positive constant \( C \) such that \( Bf(z) \leq Cf(z) \) for all \( z \in \mathbb{D} \), where \( Bf \) is a Berezin transform of \( f \).

Here and elsewhere constants are denoted by \( C \) which are positive and may be different from one occurrence to the next.

Let \( \beta(z, w) \) denote the Bergman metric between two points \( z \) and \( w \) in \( \mathbb{D} \). It is well known that

\[
\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}.
\]
For \( z \in \mathbb{D} \) and \( R > 0 \) we use
\[
\mathbb{D}(z, R) = \{ w \in \mathbb{D} : \beta(z, w) < R \}
\]
to denote the Bergman metric ball at \( z \) with radius \( R \). If \( R \) is fixed, then it can be checked that the area of \( \mathbb{D}(z, R) \), denoted by \( |\mathbb{D}(z, R)| \), is comparable to \((1 - |z|^2)^2\) as \( z \) approaches the unit circle (see \[8\]).

Fix a positive \( r \) and denote by
\[
\hat{f}_r(z) = \frac{1}{|\mathbb{D}(z, r)|} \int_{\mathbb{D}(z, r)} f(w) dA(w)
\]
the average of \( f \) over the Bergman metric ball \( \mathbb{D}(z, r) \). We define the mean oscillation of \( f \) at \( z \) in the Bergman metric to be (see \[8\])
\[
MO_r(f)(z) = \left\{ \frac{1}{|\mathbb{D}(z, r)|} \int_{\mathbb{D}(z, r)} |f(w) - \hat{f}_r(z)|^2 dA(w) \right\}^{1/2}.
\]

It is easy to verify that for any \( z \in \mathbb{D} \),
\[
[MOR_r(f)(z)]^2 = [\hat{f}_r^2(z) - (\hat{f}_r(z))^2] = \frac{1}{2|\mathbb{D}(z, r)|^2} \int_{\mathbb{D}(z, r)} \int_{\mathbb{D}(z, r)} |f(u) - f(v)|^2 dA(u) dA(v).
\]

Given a function \( f \in L^2_\beta(\mathbb{D}, dA) \), define
\[
MO(f)(z) = \left( \int_{\mathbb{D}} |f \circ \varphi(z) - f(z)|^2 dA(w) \right)^{1/2}.
\]
We call \( MO(f)(z) \) as the invariant mean oscillation of \( f \) in the Bergman metric at the point \( z \), since we have
\[
MO(f \circ \varphi)(z) = MO(f)(\varphi(z)),
\]
where \( \varphi \in \text{Aut}(\mathbb{D}) \), the group of Möbius maps of the unit disk \( \mathbb{D} \).

The main results in \[6\] can be stated as follows:

**Theorem 2.2.** Let \( K \) satisfy condition (4) and \( r > 0 \). Then the following statements are equivalent:

1. \( f \in Q_K \);
2. \( \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} [MO(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) < \infty \);
3. \( \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} [MOR(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) < \infty \);
4. \( \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - z\overline{w}|^2} K(1 - |\varphi_a(z)|^2) dA(z) dA(w) < \infty \).

**Theorem 2.3.** Let \( K \) satisfy condition (4) and \( r > 0 \). Then the following statements are equivalent:

1. \( f \in Q_{K,0} \);
2. \( \lim_{|a| \to 1} \int_{\mathbb{D}} [MO(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) = 0 \);
(3) \( \lim_{|a| \to 1} \int_D |MO_r(f)(z)|^2 K(1 - |\varphi_a(z)|^2) dr(z) = 0; \)

(4) \( \lim_{|a| \to 1} \int_D \frac{|f(z) - f(w)|^2}{|1 - z\bar{w}|^4} K(1 - |\varphi_a(z)|^2) dA(z) dA(w) = 0. \)

For a sub-arc \( I \) of \( \partial \mathbb{D} \), \(|I|\) is the length of \( I \) and
\[
S(I) = \{ r\zeta : \zeta \in I, 1 - |I| < r < 1 \}
\]
is the corresponding Carleson square.

A positive Borel measure \( \mu \) on \( \mathbb{D} \) is called a vanishing \( K \)-Carleson measure if
\[
\sup_I \int_{S(I)} K \left( \frac{1 - |z|}{|I|} \right) d\mu(z) < \infty,
\]
where the supremum is taken over all sub-arcs \( I \subset \partial \mathbb{D} \).

A positive Borel measure \( \mu \) on \( \mathbb{D} \) is called a vanishing \( K \)-Carleson measure if
\[
\lim_{|I| \to 0} \int_{S(I)} K \left( \frac{1 - |z|}{|I|} \right) d\mu(z) = 0.
\]

See [4] for more results on the \( K \)-Carleson measure.

**Lemma 2.4.** Suppose \( K \) satisfies the following two conditions:

(a) There exists a constant \( C > 0 \) such that \( K(2t) \leq CK(t) \) for all \( t > 0 \).

(b) The auxiliary function \( \varphi_K \) has the property that
\[
\int_0^1 \varphi_K(s) \frac{ds}{s} < \infty.
\]

By [4] we know that a positive Borel measure \( \mu \) on \( \mathbb{D} \) is a \( K \)-Carleson measure if and only if
\[
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} K(1 - |\varphi_a(z)|^2) d\mu(z) < \infty.
\]

Using Lemma 2.4, the second author and Zhu gave some characterizations of \( Q_K \) and \( Q_{K,0} \) spaces in terms of \( K \)-Carleson and vanishing \( K \)-Carleson measures, respectively (see [6]).

3. THE SPACES \( Q_K \)

For a function \( f \) on \( \mathbb{D} \), the function \( w_r(f)(z) \) on \( \mathbb{D} \) defined by
\[
w_r(f)(z) = \sup_{w \in \mathbb{D}(z,r)} |f(z) - f(w)|
\]
is called the oscillation of \( f \) at \( z \) in the Bergman metric (see [8]). Similarly, define another oscillation of \( f \) at \( z \) in the Bergman metric as follows:
\[
\tilde{w}_r(f)(z) = \sup_{w \in \mathbb{D}(z,r)} |\tilde{f}_r(z) - f(w)|.
\]
For $1 < p < \infty$ write
\[
O_1(f)(z) = \left\{ \frac{1}{|D(z, r)|} \int_{D(z, r)} |f(w) - f(z)|^p dA(w) \right\}^{1/p}
\]
and
\[
O_2(f)(z) = \left\{ \frac{1}{|D(z, r)|} \int_{D(z, r)} |f(w) - \hat{f}_r(z)|^p dA(w) \right\}^{1/p}.
\]

The main result in this note is the following.

**Theorem 3.1.** Let $K$ satisfy condition (4) and $r > 0$. Then the following statements are equivalent for all functions $f \in H(D)$.

1. $f \in Q_K$;
2. $\sup_{a \in \mathbb{D}} \int_D |w_r(f)(z)|^2 K(1 - |\varphi_a(z)|^2) d\tau(z) < \infty$;
3. $\sup_{a \in \mathbb{D}} \int_D |\hat{w}_r(f)(z)|^2 K(1 - |\varphi_a(z)|^2) d\tau(z) < \infty$;
4. $\sup_{a \in \mathbb{D}} \int_D [O_1(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) < \infty$;
5. $\sup_{a \in \mathbb{D}} \int_D [O_2(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z) < \infty$.

**Proof.** The proof will follow by the routes (2) $\Rightarrow$ (3) $\Rightarrow$ (1) $\Rightarrow$ (4) and (4) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). Since $(1 - |z|^2) \sim (1 - |w|^2)$ for $w \in D(z, r)$, we have
\[
\sup_{w \in D(z, r)} |\hat{f}_r(z) - f(w)| \leq C \sup_{u \in D(z, r)} \frac{1}{|D(z, r)|} \int_{D(z, r)} |f(w) - f(u)| dA(u)
\]
\[
\leq C \sup_{u \in D(z, r)} \sup_{w \in D(z, r)} |f(w) - f(u)|
\]
\[
\leq C \sup_{u \in D(z, r)} \sup_{w \in D(z, r)} (|f(w) - f(z)| + |f(z) - f(u)|)
\]
\[
\leq C \left( \sup_{u \in D(z, r)} |f(w) - f(z)| + \sup_{u \in D(z, r)} |f(z) - f(u)| \right)
\]
\[
= C \sup_{u \in D(z, r)} |f(w) - f(z)|.
\]
Hence, there exists a constant $C$ such that
\[
\hat{w}_r(f)(z) \leq C w_r(f)(z).
\]
It follows that for each $a \in \mathbb{D}$ the integral
\[
\int_D [\hat{w}_r(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z)
\]
is less than or equal to $C$ times the integral
\[
\sup_{a \in \mathbb{D}} \int_D [w_r(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z).
\]
This shows that the condition (2) implies (3). (3) ⇒ (1). A simple calculation shows that
\[ |g'(0)|^2 \leq C \int_{D(0,r)} |g(u)|^2 dA(u) \]
holds for all \( g \in H(D) \), where \( C \) is a positive constant depending on \( r \) only.
Replacing \( g = f \circ \varphi_z - \hat{f}_r(z) \) and using the fact that \( (1 - |z|^2) \) is comparable to \( |1 - \overline{w} w| \) and \( |D(z, r)|^{1/2} \) when \( w \in D(z, r) \), we get
\[
(1 - |z|^2)|f'(z)|^2 \leq C \int_{D(0,r)} |f(u) - \hat{f}_r(z)|^2 \frac{(1 - |z|^2)^2}{1 - |z|^2} dA(u)
\]
\[
\leq C \int_{D(z,r)} |f(u) - \hat{f}_r(z)|^2 \frac{1 - |z|^2}{|1 - \overline{w} w|^4} dA(u)
\]
\[
\leq C \sup_{w \in D(z,r)} |f(w) - \hat{f}_r(z)|^2.
\]

Therefore,
\[
\sup_{a \in D} \int_{D(z,r)} |f'(z)|^2 K(1 - |\varphi_a(z)|^2) dA(z)
\]
\[
\leq C \sup_{a \in D} \int_{D} [\hat{w}_r(f)(z)]^2 K(1 - |\varphi_a(z)|^2) d\tau(z).
\]
This completes that (3) implies (1).

(1) ⇒ (4). For \( w \in D(z, r) \), we have (see [8])
\[
\frac{1}{|D(z, r)|} \sim \frac{(1 - |z|^2)^2}{|1 - \overline{w} w|^4} \sim \frac{1}{(1 - |z|^2)^2}.
\]
By making a change of variables, there exists a constant \( C > 0 \) such that
\[
\left( \frac{1}{|D(z, r)|} \int_{D(z,r)} |f(w) - f(z)|^p dA(w) \right)^{\frac{1}{p}}
\]
\[
\leq C \left( \int_{D(z,r)} |f(w) - f(z)|^p \frac{(1 - |z|^2)^2}{|1 - \overline{w} w|^4} dA(w) \right)^{\frac{1}{p}}
\]
\[
= C \left( \int_{D(0,r)} |f \circ \varphi_z (w) - f \circ \varphi_z (0)|^p dA(w) \right)^{\frac{1}{p}}
\]
\[
\leq C \left( \int_{D(0,r)} (1 - |w|^2)^p |f \circ \varphi_z '(w)|^p dA(w) \right)^{\frac{1}{p}}
\]
\[ C \left( \int_{D(z,r)} (1 - |\varphi(z)|^2)^p |f' \circ \varphi(z)|^p dA(w) \right)^{\frac{1}{p}} \]

\[ = C \left( \int_{D(z,r)} (1 - |w|^2)^p |f'(w)|^p \left( \frac{1 - |z|^2}{1 - z\bar{w}} \right)^{\frac{1}{4}} dA(w) \right)^{\frac{1}{p}} \]

\[ \leq C \sup_{w \in D(z,r)} (1 - |w|^2)^p |f'(w)| \left( \int_{D(z,r)} \frac{(1 - |z|^2)^2}{(1 - z\bar{w})^4} dA(w) \right)^{\frac{1}{p}} \]

\[ \leq C (1 - |w|^2)^p |f'(w)|. \]

Hence, (1) implies (4).

(4) \Rightarrow (2). Since \(|g(v)|^p\) is subharmonic for all functions \(g \in H(D)\) for \(0 < p < +\infty\),

\[ |g(w)|^p \leq \frac{C}{|D(w, r)|} \int_{D(w, r)} |g(v)|^p dA(v). \]

Replacing \(g\) by \(f - f(z)\), we get

\[ |f(w) - f(z)|^p \leq \frac{C}{|D(w, r)|} \int_{D(w, r)} |f(u) - f(z)|^p dA(u). \]

For \(w \in D(z, r)\), we have \(D(w, r) \subset D(z, 2r)\) and there is a constant \(N > 0\) such that

\[ \frac{C}{|D(w, r)|} \leq \frac{N}{|D(z, 2r)|}. \]

Therefore,

\[ \sup_{w \in D(z,r)} |f(w) - f(z)| \leq \left( \frac{C}{|D(z, 2r)|} \int_{D(z, 2r)} |f(w) - f(z)|^p dA(w) \right)^{\frac{1}{p}} \leq CO_1(f). \]

This gives (2) by (4). Carefully checking the above proof, we can see that (5) is equivalent to the other 4 conditions. \(\square\)

Similarly to Theorems 10 and 12 in [6], we have the following \(K\)-Carleson measure characterizations of \(Q_K\) spaces.

**Theorem 3.2.** Let \(K\) satisfy condition (4) and the conditions (a) and (b) in Lemma 2.4 and \(r > 0\). Then the following statements are equivalent for all functions \(f \in H(D)\).

1. \(f \in Q_K;\)
2. \(d\mu(z) = [w_r(f)(z)]^2 d\tau(z)\) is a \(K\)-Carleson measure;
3. \(d\mu(z) = [\hat{w}_r(f)(z)]^2 d\tau(z)\) is a \(K\)-Carleson measure;
4. \(d\mu(z) = [O_1(f)(z)]^2 d\tau(z)\) is a \(K\)-Carleson measure;
5. \(d\mu(z) = [O_2(f)(z)]^2 d\tau(z)\) is a \(K\)-Carleson measure.
4. The spaces $Q_{K,0}$

Since our earlier estimates are pointwise estimates with respect to $a \in \mathbb{D}$, we have the corresponding “little o” version characterizations of $Q_{K,0}$ spaces. We omit the details of the proofs.

**Theorem 4.1.** Let $K$ satisfy the condition (4) and $r > 0$. Then the following statements are equivalent for all functions $f \in H(\mathbb{D})$.

1. $f \in Q_{K,0}$;
2. $\lim_{|a| \to 1} \int_{\mathbb{D}} |w_r(f)(z)|^2 K(1 - |\varphi_a(z)|^2) d\tau(z) = 0$;
3. $\lim_{|a| \to 1} \int_{\mathbb{D}} |\hat{w}_r(f)(z)|^2 K(1 - |\varphi_a(z)|^2) d\tau(z) = 0$;
4. $\lim_{|a| \to 1} \int_{\mathbb{D}} |O_1(f)(z)|^2 K(1 - |\varphi_a(z)|^2) d\tau(z) = 0$;
5. $\lim_{|a| \to 1} \int_{\mathbb{D}} |O_2(f)(z)|^2 K(1 - |\varphi_a(z)|^2) d\tau(z) = 0$.

**Theorem 4.2.** Let $K$ satisfy the condition (4) and the conditions (a) and (b) in Lemma 2.4 and $r > 0$. Then the following statements are equivalent for all functions $f \in H(\mathbb{D})$.

1. $f \in Q_{K,0}$;
2. $d\mu(z) = |w_r(f)(z)|^2 d\tau(z)$ is a vanishing $K$–Carleson measure;
3. $d\mu(z) = |\hat{w}_r(f)(z)|^2 d\tau(z)$ is a vanishing $K$–Carleson measure;
4. $d\mu(z) = |O_1(f)(z)|^2 d\tau(z)$ is a vanishing $K$–Carleson measure;
5. $d\mu(z) = |O_2(f)(z)|^2 d\tau(z)$ is a vanishing $K$–Carleson measure.

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