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Finite products of probabilistic normed spaces

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Abstract. We consider finite products of probabilistic normed spaces. As is to be expected, the dominance relation plays a central role.

1. Introduction

In this note we consider products of probabilistic normed spaces. We assume that the reader is familiar with the basics of the theory of probabilistic metric spaces [11]. However, in order to make this note essentially self-contained, we recall the following:

Definition 1. A *triangular norm* (briefly, a *t–norm*) is a mapping T from I^2, the closed unit square, to I, the closed unit interval, such that for all a, b, c, d in I,
(1.) $T(a, 1) = a$,
(2.) $T(a, b) = T(b, a)$,
(3.) $T(a, b) \leq T(c, d)$ whenever $a \leq c$, $b \leq d$,
(4.) $T(T(a, b), c) = T(a, T(b, c))$.

Definition 2. An *s–norm* is a function S from I^2 to I satisfying the conditions (2), (3), (4) and the boundary condition

$$S(a, 0) = a \text{ for all } a \text{ in } I.$$

If T is a t–norm, then the function T^* defined on I^2 by

$$T^*(a, b) = 1 - T(1 - a, 1 - b)$$

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is an s–norm which we refer to as the *t–conorm* of T. In particular, the function $M : I^2 \to I$ given by

$$M(a, b) = Min(a, b)$$

is a t–norm whose t–conorm is the function $M^* : I^2 \to I$ given by

$$M^*(a, b) = Max(a, b).$$

In the sequel, unless stated otherwise, we will assume that all t–norms and s–norms are continuous.

Let, as usual, Δ^+ denote the set of all one–dimensional probability distributions whose support is the positive half-line, i.e., Δ^+ is the set of all functions such that Dom $F = [0, +\infty]$, Ran $F \subseteq I$, $F(0) = 0, F(+\infty) = 1$, and F is non–decreasing and left–continuous on $(0, +\infty)$. The set Δ^+ is ordered by the usual pointwise ordering of functions; and ε_0 is the function in Δ^+ given by

$$\varepsilon_0(x) := \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

Definition 3. A *triangle function* is a mapping τ from $\Delta^+ \times \Delta^+$ into Δ^+ such that for all F, G, H, K in Δ^+,

1. $\tau(F, \varepsilon_0) = F$,
2. $\tau(F, G) = \tau(G, F)$,
3. $\tau(F, G) \leq \tau(H, K)$ whenever $F \leq H$, $G \leq K$,
4. $\tau(\tau(F, G), H) = \tau(F, \tau(G, H))$.

Particular triangle functions are the functions τ_T, τ_{T^*} and Π_T which, for any continuous t–norm T, and any $x \geq 0$, are given by

$$\tau_T(F, G)(x) = \sup\{T(F(u), G(v)) \mid u + v = x\},$$
$$\tau_{T^*}(F, G)(x) = \inf\{T^*(F(u), G(v)) \mid u + v = x\}$$

and

$$\Pi_T(F, G)(x) = T(F(x), G(x)).$$

Definition 4. A *probabilistic normed space*, briefly a PN space, is a quadruple (V, ν, τ, τ^*) in which V is a linear space, τ and τ^* are continuous triangle functions with $\tau \leq \tau^*$ and ν, the probabilistic norm, is a map $\nu : V \to \Delta^+$ such that
(N1) $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$, θ being the null vector in V;
(N2) $\nu_{-p} = \nu_p$ for every $p \in V$;

(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$ for all $p, q \in V$;
(N4) $\nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$ for every $\alpha \in [0,1]$ and for every $p \in V$.

If only (N1) and (N2) hold, then we say that the pair (V, ν) is a *probabilistic semi–normed* (briefly PSN) space.

If, instead of (N1), we only have $\nu_\theta = \varepsilon_0$, then we shall speak of a *probabilistic pseudo–normed space*, briefly a PPN space. If the inequality (N4) is replaced by the equality $\nu_p = \tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p})$, then the PN space is called a *Šerstnev space* and, as a consequence, a condition stronger than (N2) holds, namely, for all $\lambda \neq 0$ and all p in V,

$$\nu_{\lambda p} = \nu_p \left(\frac{j}{|\lambda|} \right).$$

Here j is the identity map on $\mathbf{R}$. A Šerstnev space is denoted by (V, ν, τ).

Definition 5. A *Menger PN space* is a PN space (V, ν, τ, τ^*) in which $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$ for some t–norm T and its t–conorm T^*.

2. The dominance relation

Definition 6. Let $(S, \leq)$ be a partially ordered set and let f and g be commutative and associative binary operations on S with common identity e. Then f *dominates* g, and we write $f \gg g$, if for all $x_1, x_2, y_1, y_2 \in S$,

$$f(g(x_1, y_1), g(x_2, y_2)) \geq g(f(x_1, x_2), f(y_1, y_2)).$$

Setting $y_1 = x_2 = e$ in this inequality, one has $f(x_1, y_2) \geq g(x_1, y_2)$, whence $f \gg g$ implies $f \geq g$, which in turn implies that the dominance relation is antisymetric, and it is easy to show that the relation is also reflexive. However, as a simple example due to H. Sherwood shows, in general it is not transitive [4].

We are interested in the dominance relation as it applies to t–norms, s–norms and triangle functions. Here the following are known:

Lemma 1. *The following statements hold:*
(a) *For any t-norm T, $M \gg T$.*
(b) *For any s-norm S, $S \gg M^*$.*
(c) *For any continuous t–norm T, $\Pi_T \gg \tau_T$.*
(d) *For any triangle function τ, $\Pi_M \gg \tau$.*

Proof. (a) For any $x_1, x_2, y_1, y_2 \in I$, we have

$$x_1 \geq M(x_1, x_2), \qquad y_1 \geq M(y_1, y_2),$$

thus

$$T(x_1, y_1) \geq T(M(x_1, x_2), M(y_1, y_2)).$$

Similarly,

$$T(x_2, y_2) \geq T(M(x_1, x_2), M(y_1, y_2)),$$

whence

$$M(T(x_1, y_1), T(x_2, y_2)) \geq T(M(x_1, x_2), M(y_1, y_2))$$

i.e. $M \gg T$. Corresponding arguments prove (b),(c) and (d).

Next, a straightforward calculation yields:

Lemma 2. *For any t–norms T_1 and T_2, if $T_1 \gg T_2$ then $T_2^* \gg T_1^*$, and conversely.*

The following theorem is due to R.M. Tardiff (see [13]):

Theorem 1. *For any continuous t-norms T_1 and T_2, the following are equivalent:*
(1) $T_1 \gg T_2$,
(2) $\Pi_{T_1} \gg \Pi_{T_2}$,
(3) $\tau_{T_1} \gg \tau_{T_2}$,
(4) $\Pi_{T_1} \gg \tau_{T_2}$,
(5) $\tau_{T_2^*} \gg \tau_{T_1^*}$.

It is also known that the dominance relation is transitive on certain subsets of the set of continuous t–norms (see [6], [12], [14]). But whether it is transitive on the set of all t–norms is still an open question.

3. Finite products of PN spaces

Definition 7. Let (V_1, ν_1) and (V_2, ν_2) be PSN spaces and let τ be a triangle function. Then their τ–product is the pair $(V_1 \times V_2, \nu_\tau)$, where

$$\nu_\tau : V_1 \times V_2 \to \Delta^+$$

is given by

$$\nu_\tau((p_1, p_2)) = \tau(\nu_1(p_1), \nu_2(p_2)).$$

Clearly $(V_1 \times V_2, \nu_\tau)$ is a PSN space.

Theorem 2. *Let* $(V_1, \nu_1, \tau, \tau^*)$, $(V_2, \nu_2, \tau, \tau^*)$ *be PN spaces under the same triangle functions* τ *and* τ^* *and suppose that there is a triangle function* σ *such that* $\tau^* \gg \sigma$ *and* $\sigma \gg \tau$. *Then their* σ*–product is a PN space under* τ *and* τ^*.

Proof. Let $\bar{p} = (p_1, p_2)$ and $\bar{q} = (q_1, q_2)$ be points in $(V_1 \times V_2)$. Then, since $\sigma \gg \tau$ we have

$$\begin{aligned}\nu_\sigma(\bar{p} + \bar{q}) &= \sigma(\nu_1(p_1 + q_1), \nu_2(p_2 + q_2))\\ &\geq \sigma(\tau(\nu_1(p_1), \nu_1(q_1)), \tau(\nu_2(p_2), \nu_2(q_2)))\\ &\geq \tau(\sigma(\nu_1(p_1), \nu_2(p_2)), \sigma(\nu_1(q_1), \nu_2(q_2)))\\ &= \tau(\nu_\sigma(\bar{p}), \nu_\sigma(\bar{q})).\end{aligned}$$

Next, for any α in I, we have

$$\nu_1(p_1) \leq \tau^*(\nu_1(\alpha p_1), \nu_1(1-\alpha)p_1)$$

and

$$\nu_2(p_2) \leq \tau^*(\nu_2(\alpha p_2), \nu_2(1-\alpha)p_2)$$

whence, since $\tau^* \gg \sigma$ one has

$$\begin{aligned}\nu_\sigma(\bar{p}) &= \sigma(\nu_1(p_1), \nu_2(p_2))\\ &\leq \sigma(\tau^*(\nu_1(\alpha p_1), \nu_1((1-\alpha)p_1)), \tau^*(\nu_2(\alpha p_2), \nu_2((1-\alpha)p_2)))\\ &\leq \tau^*(\sigma(\nu_1(\alpha p_1), \nu_2(\alpha p_2)), \sigma(\nu_1((1-\alpha)p_1), \nu_2((1-\alpha)p_2)))\\ &= \tau^*(\nu_\sigma(\alpha\bar{p}), \nu_\sigma((1-\alpha)\bar{p})).\end{aligned}$$

Example 1. The Π_T–product $(V_1 \times V_2,\, \nu_{\Pi_T})$ of the PN spaces $(V_1,\, \nu_1,\, \tau_T,\, \Pi_M)$ and $(V_2,\, \nu_2, \tau_T,\, \Pi_M)$ is a PN space under τ_T and Π_M.

Example 2. Let $(V_1, F,\, \Pi_M)$ and $(V_2, G,\, \Pi_M)$ be equilateral PN spaces with distribution functions F, G respectively. Then, their Π_M–product is an equilateral PN space with d.f. given by $\Pi_M(F,\, G)$.

In particular, if $F \equiv G$, the Π_M–product is an equilateral PN space with the same distribution function F.

Corollary 1. *If* $\tau^* \gg \tau$, *then the* τ^**–product, as well as the* τ*–product of* $(V_1, \eta_1, \tau, \tau^*)$ *and* $(V_2, \eta_2, \tau, \tau^*)$ *is a PN space under* τ *and* τ^*.

Corollary 2. *If* (V_1, ν_1, τ) *and* (V_2, ν_2, τ) *are PN spaces in the sense of Šerstnev and* $\tau_M \gg \tau$, *then their* τ_M*–product, as well as their* τ*–product, is also a PN space in the sense of Šerstnev.*

In the case of Menger spaces we have a more interesting result.

Corollary 3. *If* (V_1, ν_1, T) *and* (V_2, ν_2, T) *are Menger PN spaces under the same continuous t–norm* T*, then their* τ_M*–product is also a Menger PN space under* T*.*

Proof. Since, for any t–norm T, $M \gg T$ and $T^* \gg M^*$, by Theorem 2 we have $\tau_M \gg \tau_T$ and $\tau_{T^*} \gg \tau_{M^*}$.

But, as is well-known, $\tau_M = \tau_{M^*}$, whence the conclusion follows. The above results clearly extend to products of a finite number of PN spaces.

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Konačni produkti vjerovatnostnih normiranih prostora

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Sadržaj

U radu se razmatraju konačni produkti vjerovatnostnih normiranih prostora. Kao što je bilo i za očekivati, relacija dominacije igra glavnu ulogu.