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On $K-$contact $\eta-$Einstein manifolds

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Abstract. The object of the present paper is to study a K–contact $\eta-$Einstein manifold satisfying a certain condition on the curvature tensor.

1. Introduction

Let (M^n, g) be a contact Riemannian manifold with contact form η, associated vector field ξ, $(1,1)$–tensor field ϕ and associated Riemannian metric g. If ξ is a killing vector field, then M^n is called a *K-contact Riemannian manifold* [1],[2]. A K–contact Riemannian manifold is called *Sasakian* [1], if the relation

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X \tag{1}$$

holds, where ∇ denotes the covariant differentiation operator with respect of g.

Recently, M.C. Chaki and M. Tarafdar [3] studied a Sasakian manifold M^n $(n > 3)$ satisfying the relation $R(X,Y).C = 0$, where $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold and C is the Weyl conformal curvature tensor of type $(1,3)$. Generalizing the result of Chaki and Tarafdar [3], N. Guha and U.C. De [4] proved that if a K–contact manifold with characteristic vector field ξ belonging to the $K-$nullity distribution satisfies the condition $R(\xi,X).C = 0$, then $C(\xi,X)Y = 0$ for any vector fields X, Y. Shaikh Abbos Ali, U.C. De and T.Q. Binh proved the following theorem.

Theorem 1. [7] *A K–contact $\eta-$Einstein manifold (M^n, g) $(n > 3)$ satisfying the condition $R(X,\xi).C = 0$ is a space of constant curvature 1.*

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2. Preliminaries

In a $K-$contact Riemannian manifold the following relations hold ([1], [2], [5]):

$$\text{a) } \phi\xi = 0, \quad \text{b) } \eta(\xi) = 1, \quad \text{c) } g(X,\xi) = \eta(X) \tag{2}$$

$$\phi^2 X = -X + \eta(X)\xi \tag{3}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{4}$$

$$\nabla_X \xi = -\phi X \tag{5}$$

$$g(R(\xi,X)Y,\xi) = \eta(R(\xi,X)Y) = g(X,Y) - \eta(X)\eta(Y) \tag{6}$$

$$R(\xi,X)\xi = -X + \eta(X)\xi \tag{7}$$

$$S(X,\xi) = (n-1)\eta(X) \tag{8}$$

$$(\nabla_X \phi)Y = R(\xi,X)Y \tag{9}$$

for any vector fields X, Y.

A K–contact manifold M^n is said to be $\eta-$*Einstein* if its Ricci tensor S is of the form $S = ag + b\eta \otimes \eta$, where a, b are smooth functions on M.

In this case we have

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y). \tag{10}$$

Putting $X = Y = \xi$ in (10) and then using (8) and (2)b, we get

$$a + b = (n-1). \tag{11}$$

Also (10) implies that

$$\tau = an + b. \tag{12}$$

From (11) and (12) we have

$$a = \frac{\tau}{n-1} - 1, \quad b = n - \frac{\tau}{n-1}. \tag{13}$$

Again from (10) we obtain

$$QX = (\frac{\tau}{n-1} - 1)X + (n - \frac{\tau}{n-1})\eta(X)\xi \tag{14}$$

where Q denotes the Ricci operator, i.e. $S(X,Y) = g(QX,Y)$.

By definition the Weyl conformal curvature tensor C is given by

$$\begin{aligned} C(X,Y)Z = {} & R(X,Y)Z \\ & - \frac{1}{n-2}\left[g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y\right] \\ & + \frac{\tau}{(n-1)(n-2)}\left[g(Y,Z)X - g(X,Z)Y\right] \end{aligned} \tag{15}$$

Using (10) and (14) in (15), we get

$$
\begin{aligned}
C(X,Y)Z = R(X,Y)Z& \\
&+\left(\frac{2}{(n-2)}-\frac{\tau}{(n-1)(n-2)}\right)[g\,(Y,Z)X-g\,(X,Z)Y] \\
&-\left(\frac{n}{(n-2)}-\frac{\tau}{(n-1)(n-2)}\right)[g\,(Y,Z)\eta(X)\xi-g\,(X,Z)\eta(Y)\xi \\
&\qquad\qquad+\eta(Z)\eta(Y)X-\eta(Z)\eta(X)Y] \quad (16)
\end{aligned}
$$

The endomorphisms $X\wedge Y$ and $X\wedge_S Y$ and the homeomorphisms $R(X,\xi).C$ and $C(X,\xi).R$ are defined by

$$(X\wedge Y)Z = g(Y,Z)X-g(X,Z)Y, \tag{17}$$

$$(X\wedge_S Y)Z = S(Y,Z)X-S(X,Z)Y, \tag{18}$$

$$
\begin{aligned}
(R(X,\xi).C)(U,V)W =&R(X,\xi)C(U,V)W-C(R(X,\xi)U,V)W \\
&-C(U,R(X,\xi)V)W-C(U,V)R(X,\xi)W, \quad (19)\\
(C(X,\xi).R)(U,V)W =&C(X,\xi)R(U,V)W-R(C(X,\xi)U,V)W \\
&-R(U,C(X,\xi)V)W-R(U,V)C(X,\xi)W \quad (20)
\end{aligned}
$$

respectively, where X,Y,Z are vector fields of M.

3. Main results

Theorem 2. *A K–contact $\eta-$Einstein manifold (M^n,g) $(n>3)$ satisfying the condition $R(X,\xi).C=C(X,\xi).R$ is space of constant curvature* 1.

Proof. From (19) by definition we have

$$
\begin{aligned}
(R(X,\xi).C)(U,V)W =&R(X,\xi)C(U,V)W-C(R(X,\xi)U,V)W \\
&-C(U,R(X,\xi)V)W-C(U,V)R(X,\xi)W \quad (21)
\end{aligned}
$$

for all vector fields X,U,V,W.

Substituting U and W by ξ in (21) yields

$$
\begin{aligned}
(R(X,\xi).C)(\xi,V)\xi =&R(X,\xi)C(\xi,V)\xi-C(R(X,\xi)\xi,V)\xi \\
&-C(\xi,R(X,\xi)V)\xi-C(\xi,V)R(X,\xi)\xi \quad (22)
\end{aligned}
$$

From (16) we get by virtue of (2)b, (2)c and (7),

$$C(\xi,V)\xi = C(V,\xi)\xi = 0, \quad \text{for any vector field } V. \tag{23}$$

Thus we have

$$(R(X,\xi).C)(\xi,V)\xi = -C(\xi,R(X,\xi)V)\xi - C(\xi,V)R(X,\xi)\xi. \tag{24}$$

Using (7) we get

$$C(R(X,\xi)\xi,V)\xi = C(X,V)\xi \tag{25}$$
$$C(\xi,V)R(X,\xi)\xi = C(\xi,V)X. \tag{26}$$

Thus we have

$$(R(X,\xi).C)(\xi,V)\xi = -C(X,V)\xi - C(\xi,V)X. \tag{27}$$

On the other hand

$$\begin{aligned}(C(X,\xi).R)(\xi,V)\xi =& C(X,\xi)R(\xi,V)\xi - R(C(X,\xi)\xi,V)\xi \\ & - R(\xi,C(X,\xi)V)\xi - R(\xi,V)C(X,\xi)\xi.\end{aligned} \tag{28}$$

Using (6) and (23) we obtain the following equations

$$C(X,\xi)R(\xi,V)\xi = -C(X,\xi)V \tag{29}$$
$$R(C(X,\xi)\xi,V)\xi = 0 \tag{30}$$
$$R(\xi,C(X,\xi)V)\xi = -C(X,\xi)V \tag{31}$$
$$R(\xi,V)C(X,\xi)\xi = 0. \tag{32}$$

Using (29), (30), (31) and (32) in (28) we obtain

$$(C(X,\xi).R)(\xi,V)\xi = 0.$$

Thus our condition satisfies the following equation

$$(R(X,\xi).C)(\xi,V)\xi = 0.$$

By a similar discussions in the proof of the Theorem in [7], we obtain

$$R(X,V)\xi = \eta(V)X - \eta(X)V \tag{33}$$

from which it follows that

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X. \tag{34}$$

In view of (1), (9) and (34), we obtain that the manifold is Sasakian and hence by the result of Chaki and Tarafdar [3], the manifold is a space of constant curvature 1. □

A contact Riemannian manifold satisfying the condition $R(X,\xi).C = 0$ have been studied by C. Baikoussis and T. Koufogiorgos [6].

Theorem 3. *A $K-$contact $\eta-$Einstein manifold (M^n, g) $(n > 3)$ satisfying the condition $R(X,\xi).C = L\{(X \wedge \xi).C\}$, $(L \neq 1)$ is space of constant curvature, where L is some function on M^n.*

Proof. We denote the expression in the bracket on the right-hand side of (19) by A, and we calculate it. Thus

$$\begin{aligned} A =& L\{((X \wedge \xi).C)(\xi, V)\xi\} = L\{(X \wedge \xi)C(\xi, V)\xi \\ & - C((X \wedge \xi)\xi, V)\xi - C(\xi, (X \wedge \xi)V)\xi - C(\xi, V)(X \wedge \xi)\xi\}. \end{aligned} \tag{35}$$

Using (23) we have

$$(X \wedge \xi)C(\xi, V)\xi = 0.$$

$$\begin{aligned} C((X \wedge \xi)\xi, V)\xi &= C(X - \eta(X)\xi, V)\xi \\ &= C(X, V)\xi - \eta(X)C(\xi, V)\xi \\ &= C(X, V)\xi. \end{aligned}$$

$$\begin{aligned} C(\xi, (X \wedge \xi)V)\xi &= 0 \\ C(\xi, V)(X \wedge \xi)\xi &= C(\xi, V)(X - \eta(X))\xi. \end{aligned}$$

From the above and using (27) we have

$$\begin{aligned} -C(X, V)\xi - C(\xi, V)X &= L\{-C(X, V)\xi - C(\xi, V)X\} \\ (L - 1)\{C(X, V)\xi + C(\xi, V)X\} &= 0 \end{aligned} \tag{36}$$

$$(L - 1)\{R(X, V)\xi + R(\xi, V)X - g(X, V) + 2\eta(X)V - \eta(V)X\} = 0. \tag{37}$$

Interchanging X and V in (37) we have

$$(L - 1)\{R(V, X)\xi + R(\xi, X)V - g(X, V)\xi + 2\eta(V)X - \eta(X)V\} = 0. \tag{38}$$

Subtracting (38) from (37) and then using Bianchi's first identity, we get

$$(L - 1)\{R(X, V)\xi - \eta(V)X + \eta(X)V\} = 0. \tag{39}$$

Since $L \neq 1$ we get

$$R(X, V)\xi = \eta(V)X - \eta(X)V$$

from which it follows that

$$R(\xi, X)Y = \{g(X, Y)\xi - \eta(Y)X\}. \tag{40}$$

In view of (1), (9) and (40), we obtain that the manifold is Sasakian and hence by the result of Chaki and Tarafdar [3], the manifold is a space of constant curvature 1. □

Theorem 4. *A $K-$contact $\eta-$Einstein manifold (M^n, g) $(n > 3)$ satisfying the condition $R(X, \xi).C = f\{(X \wedge_S^n \xi).C\}$, $(f \neq \frac{1}{(n-1)^n})$ is space of constant curvature, where f is some function on M^n.*

Proof. We denote the expression in the bracket on the right-hand side of (19) by A, and we calculate it. Thus

$$(R(X, \xi).C)(\xi, V)\xi = f\{((X \wedge_S^n \xi).C)(\xi, V)\xi\}. \tag{41}$$

where

$$(X \wedge_S^n Y) = S^n(Y, Z)X - S^n(X, Z)Y$$

and

$$S^n(X, Y) = g(Q^n X, Y).$$

Then

$$\begin{aligned} A =& f\{((X \wedge_S^n \xi).C)(\xi, V)\xi\} = f\{(X \wedge_S^n \xi)C(\xi, V)\xi \\ &- C((X \wedge_S^n \xi)\xi, V)\xi - C(\xi, (X \wedge_S^n \xi)V)\xi - C(\xi, V)(X \wedge_S^n \xi)\xi\}. \end{aligned} \tag{42}$$

Using (23)

$$\begin{gathered} (X \wedge_S^n \xi)C(\xi, V)\xi = 0 \\ C((X \wedge_S^n \xi)\xi, V)\xi = (n-1)^n C(X, V)\xi \\ C(\xi, (X \wedge_S^n \xi)V)\xi = 0 \\ C(\xi, V)(X \wedge_S^n \xi)\xi = (n-1)^n C(\xi, V)X. \end{gathered}$$

From the above and using (27) we have

$$-C(X, V)\xi - C(\xi, V)X = -f\{(n-1)^n C(X, V)\xi + (n-1)^n C(\xi, V)X\}$$

$$(f(n-1)^n - 1)\{C(X, V)\xi + C(\xi, V)X\} = 0 \tag{43}$$

$$(f(n-1)^n - 1)\{R(X, V)\xi + R(\xi, V)X - g(X, V)\xi + 2\eta(X)V - \eta(V)X\} = 0. \tag{44}$$

Interchanging X and V in (44) we have

$$(f(n-1)^n - 1)\{R(V, X)\xi + R(\xi, X)V - g(X, V)\xi + 2\eta(V)X - \eta(X)V\} = 0. \tag{45}$$

Subtracting (45) from (44) and then using Bianchi's first identity, we get

$$(f(n-1)^n - 1)\{R(X,V)\xi - \eta(V)X + \eta(X)V\} = 0. \tag{46}$$

Since $(f(n-1)^n - 1) \neq 0$ we get

$$R(X,V)\xi = \eta(V)X - \eta(X)V$$

from which it follows that

$$R(\xi, X)Y = \{g(X,Y)\xi - \eta(Y)X\}. \tag{47}$$

In view of (1), (9) and (47), we obtain that the manifold is Sasakian and hence by the result of Chaki and Tarafdar [3], the manifold is a space of constant curvature 1. □

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O $K-$kontakt $\eta-$Einstein mnogostrukostima

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Sadržaj

U radu se izučava $K-$kontakt $\eta-$Einstein mnogostrukost koja zadovoljava neke uslove za tenzor zakrivljenosti.