RADOVI MATEMATIČKI
Vol. 13 (2004), 93–101

On weakly Ricci symmetric spacetime manifolds

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Abstract. The object of the present paper is to study weakly Ricci symmetric spacetime manifolds. Among others it is proved that if in a weakly Ricci symmetric spacetime of non–zero constant scalar curvature the matter distribution is perfect fluid, then the acceleration vector and the expansion scalar are zero and such a spacetime can not admit heat flux. Finally a study of conformally flat perfect fluid weakly Ricci symmetric spacetime manifold is made.

1. Introduction

The present paper is concerned with certain investigations in general relativity by the coordinate free method of differential geometry. In this method of study the spacetime of general relativity is regarded as a connected four-dimensional semi–Riemannian manifold (M^4, g) with Lorentz metric g with signature $(-, +, +, +)$. The geometry of the Lorentz manifold begins with the study of the causal character of vectors of the manifold. It is due to this causality that the Lorentz manifold becomes a convenient choice for the study of general relativity.

Here we consider a special type of spacetime which is called weakly Ricci symmetric spacetime. The notion of weakly symmetric manifold was introduced by Tamassy and Binh [1]. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is called weakly symmetric if the curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)W = A(X)R(Y,Z)W + B(Y)R(X,Z)W + C(Z)R(Y,X)W \\ +D(W)R(Y,Z)X + g(R(Y,Z)W,X)\rho, \tag{1}$$

2000 Mathematics Subject Classification: Primary: 53B05, 53C15, 53B35; Secondary: 53C80, 54E70.

Keywords and phrases: Weakly Ricci symmetric manifold, weakly Ricci symmetric perfect fluid spacetime, Einstein equation, conformally flat perfect fluid $(WRS)_4$ spacetime.

where ∇ denote the Levi–Civita connection on (M^n, g) and A, B, C, D and ρ are 1-forms and a vector field respectively, which are non-zero simultaneously. Such a manifold is denoted by $(WS)_n$.

In 1993 Tamassy and Binh [2] introduced the notion of a weakly Ricci symmetric manifold. A non–flat Riemannian manifold (M^n, g) $(n > 2)$ is called weakly Ricci symmetric if its Ricci tensor S is of type (0, 2) and is not identically zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(X, Z) + D(Z)S(Y, X) \tag{2}$$

where A, B, D are three non–zero 1–forms and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Such an n–dimensional manifold was denoted by $(WRS)_n$. If in (2) the 1–form A is replaced by $2A, B$ and D are replaced by A, then the manifold is called a pseudo Ricci symmetric manifold introduced by Chaki [3]. Also if in (2) the 1–form A is replaced by $2A$, then the manifold is called a generalized pseudo Ricci symmetric manifold introduced by Chaki and Koley [4]. So the defining condition of a $(WRS)_n$ is a little weaker than that of a generalized pseudo Ricci symmetric manifold. The existence of a $(WS)_n$ is proved by M. Prvanovic [5] and a concrete example of a $(WS)_n$ is given by De and Banbyopadhyay [6] by a suitable metric. In a recent paper De and Ghosh [7] cited an example of a $(WRS)_n$. Also De and Sengupta [8] proved that if a $(WS)_n$ admits a type of semi–symmetric connection then the $(WS)_n$ reduces to

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(X, Z) + B(Z)S(X, Y) \tag{3}$$

i.e., the $(WS)_n$ reduces to a special type of $(WRS)_n$.

In the study of a $(WRS)_n$ an important role is played by the 1–form δ defined by

$$\delta(X) = B(X) - D(X).$$

Lemma 1. [9] *If* $\delta \neq 0$, *then the Ricci tensor is of the form*

$$S(X, Y) = rT(X)T(Y) \tag{4}$$

where T *is a non–zero 1–form defined by*

$$T(X) = g(X, \rho), \tag{5}$$

r *is the scalar curvature and* ρ *is called the basic vector field of* $(WRS)_n$.

It is shown that if a general relativistic perfect fluid spacetime with cosmological constant λ and flow vector field ρ is a semi–Riemannian $(WRS)_4$ with constant scalar curvature then the acceleration vector and the expansion scalar of the fluid are zero and the cosmological constant λ satisfies the inequality $-\frac{r}{6} < \lambda < \frac{3r}{2}$.

Next we prove that if in a $(WRS)_4$ spacetime the matter distribution is fluid with ρ as the velocity vector field of the fluid, then such a fluid can not admit heat flux.

Finally we consider conformally flat $(WRS)_4$ and obtain some interesting results.

2. Weakly Ricci symmetric perfect fluid spacetime

Lemma 2. *In a* $(WRS)_4$*,* $r \neq 0$ *where* r *is the scalar curvature.*

If $r = 0$ *then we have from* (4) $S = 0$ *which is inadmissible by the definition of* $(WRS)_4$*.*

A semi–Riemannian $(WRS)_4$ may similarly be defined by taking a Lorentz metric g with signature $(-, +, +, +)$. All the above relations will also hold in such $(WRS)_4$.

Now we take ρ as a timelike vector field. Then we have from (4)

$$\begin{aligned} S(X,Y) &= rT(X)T(Y) \quad \text{or} \\ S(X,\rho) &= -rT(X), \text{ since } T(\rho) = g(\rho,\rho) = -1 \quad \text{or} \\ S(X,\rho) &= -rg(X,\rho). \end{aligned} \tag{6}$$

Now we write Einstein equation as follows:

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = k[(\sigma + p)T(X)T(Y) + pg(X,Y)] \tag{7}$$

where λ is the cosmological constant, k is the gravitational constant, σ is the energy density and p is the isotropic pressure of the fluid.
Putting $Y = \rho$ in (7) we have

$$\begin{aligned} S(X,\rho) - \frac{r}{2}g(X,\rho) + \lambda g(X,\rho) &= k[(\sigma + p)T(X)T(\rho) + pg(X,\rho)] \text{ or} \\ -rT(X) - \frac{r}{2}T(X) + \lambda T(X) &= k[-(\sigma + p)T(X) + pT(X)] \text{ or} \\ -\frac{3r}{2} + \lambda &= -k\sigma \text{ or} \\ \sigma &= \frac{3r - 2\lambda}{2k}. \end{aligned} \tag{8}$$

Again contracting (7) we get

$$\begin{aligned} r - 2r + 4\lambda &= k(3p - \lambda) \text{ or} \\ -r + 4\lambda &= 3kp - \frac{3r}{2} + \lambda \text{ or} \\ -2r + 8\lambda &= 6kp - 3r + 2\lambda \text{ or} \\ 6kp &= -2r + 3r + 8\lambda - 2\lambda \text{ or} \\ p &= \frac{r + 6\lambda}{6k}. \end{aligned} \tag{9}$$

If r is a constant, then it follows from (8) and (9) that σ and p are constant. Since $\sigma > 0$, from (8) we have

$$\begin{aligned} \frac{3r - 2\lambda}{2k} &> 0 \text{ or} \\ 3r - 2\lambda &> 0 \text{ or} \\ 3r &> 2\lambda \text{ or} \\ \lambda &< \frac{3r}{2}. \end{aligned} \tag{10}$$

Since $p > 0$, similarly we have from (9)

$$\lambda > -\frac{r}{6}. \tag{11}$$

From (10) and (11) we obtain

$$-\frac{r}{6} < \lambda < \frac{3r}{2}.$$

We assume that the scalar curvature r of a $(WRS)_4$ is constant. Then from (8) and (9) we see that σ and p are both constant.

Since $\operatorname{div} T = 0$, we get the energy and force equations as follows [10]:

$$p\,\sigma = -(\sigma + p) \operatorname{div} \rho \quad \text{[Energy equation]} \tag{12}$$

$$(\sigma + p)\nabla_\rho \rho = -\operatorname{grad} p - (\rho p)\rho \quad \text{[Force equation]}. \tag{13}$$

In this case both σ and p are constant, it follows from (12) and (13) that

$$\operatorname{div} \rho = 0 \quad \text{and} \quad \nabla_\rho \rho = 0.$$

It is known that $\operatorname{div} \rho$ represents the expansion scalar and $\nabla_\rho \rho$ represents the acceleration vector. Thus in this case both the expansion scalar and the acceleration vector are zero.

Hence we can state the following:

Theorem 1. *If in a weakly Ricci symmetric spacetime of non–zero constant scalar curvature the matter distribution is perfect fluid whose velocity vector field is the basic vector field ρ of the spacetime, then the acceleration vector of the fluid must be zero and the expansion scalar also so. Moreover the cosmological constant λ satisfies the relation $-\frac{r}{6} < \lambda < \frac{3r}{2}$.*

3. Possibility of a fluid (WRS)$_4$ spacetime admitting a heat flux

In this section we discuss whether a fluid $(WRS)_4$ spacetime with the vector field ρ as the velocity vector field can admit a heat flux.

If this is possible, let the energy-momenttum tensor be of the following form:

$$T(X,Y) = (\sigma + p)T(X)T(Y) + p\, g(X,Y) + T(X)B(Y) + T(Y)B(X) \tag{14}$$

where $g(X,W) = B(X)\ \ \forall\, X$, W being the heat flux vector field, then since W is spacelike, $g(\rho, W) = 0$, or $B(\rho) = 0$.

In this case the Einstein equation can be written as

$$\begin{aligned} S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = [&(\sigma+p)T(X)T(Y) + p\, g(X,Y) \\ &+T(X)B(Y) + T(Y)B(X)]. \end{aligned} \tag{15}$$

Now from (15) we have

$$-rT(X) - \frac{r}{2}T(X) + \lambda T(X) = k[-(\sigma+p)T(X) + pT(X) - B(X)] \ \text{ or}$$

$$(-\frac{3r}{2} + \lambda + k\sigma)T(X) = -kT(X) \quad \forall\, X. \tag{16}$$

Putting $X = \rho$ in (16) we have

$$-\frac{3r}{2} + \lambda + k\sigma = 0$$

which implies

$$\begin{aligned} kB(X) &= 0 \\ \text{i.e.,}\quad B(X) &= 0, \quad \text{since } \ k \neq 0. \end{aligned} \tag{17}$$

Thus we can state the following:

Theorem 2. $(WRS)_4$ *spacetime can not admit a heat flux.*

4. Conformally flat perfect fluid (WRS)$_4$ spacetime

In this section we consider a conformally flat perfect fluid $(WRS)_4$ spacetime obeying the Einstein equation without cosmological constant and having the vector field ρ of $(WRS)_4$ as the velocity of the fluid.

It is known [11] that in a conformally flat $(WRS)_4$ the 1–form T defined by $g(X,\rho) = T(X)$ is closed, i.e., $dT(X,Y) = 0$. Hence it follows that

$$g(\nabla_X \rho, Y) = g(\nabla_Y \rho, X) \quad \forall\, X,\ Y \tag{18}$$

which means that the vector field ρ is non–rotational. Now putting $Y = \rho$ in (18) we get

$$g(\nabla_X \rho, \rho) = g(\nabla_\rho \rho, X). \tag{19}$$

Since $g(\nabla_X \rho, \rho) = 0$, from (19) it follows that

$$g(\nabla_\rho \rho, X) = 0 \quad \forall\, X.$$

Hence $\nabla_\rho \rho = 0$. This means that the integral curves of the vector field ρ are geodesics. Therefore we can state the following:

Theorem 3. *In a conformally flat* $(WRS)_4$ *spacetime the vector field* ρ *defined by* (5) *is non–rotational and its integral curves are geodesic.*

Again we have from Einsteins' equations without cosmological constant

$$\begin{aligned} S(X,Y) &= \frac{r}{2}g(X,Y) + k[(\sigma + p)T(X)T(Y) + pg(X,Y)] \\ &= \frac{r}{2}g(X,Y) + k(\frac{3r}{2k} + \frac{r}{6k})T(X)T(Y) + \frac{r}{6}g(X,Y) \\ &= \frac{2r}{3}g(X,Y) + \frac{5r}{3}T(X)T(Y) \end{aligned} \tag{20}$$

which implies

$$LX = \frac{2r}{3}X + \frac{5r}{3}T(X)\rho \tag{21}$$

where L is the symmetric endomorphism given by $S(X,Y) = g(LX,Y)$. Since the spacetime is assumed to be conformally flat we have

$$\begin{aligned} R(X,Y)Z = \frac{1}{2}[S(X,Z)X - S(X,Z)Y + g(Y,Z)LX - g(X,Z)LY] \\ -\frac{r}{6}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{22}$$

Using (20) and (21) in (22) we have

$$\begin{aligned} R(X,Y)Z = \frac{r}{2}[g(Y,Z)X - g(X,Z)Y] + \frac{5r}{6}[T(Y)T(Z)X - T(X)T(Z)Y \\ +g(Y,Z)T(X)\rho - g(X,Z)T(Y)\,\rho]. \end{aligned}$$

Let $\rho^{\perp}$ denote the 3–dimensional distribution in $(WRS)_4$ spacetime orthogonal to ρ, then

$$R(X,Y)Z = \frac{r}{2}[g(Y,Z)X - g(X,Z)Y] \quad \forall\, X,\ Y,\ \in \rho^{\perp} \tag{23}$$

and

$$R(X,\rho,\rho) = 0 \quad \text{for every} \quad X \in \rho^{\perp}. \tag{24}$$

Let $X,\ Y \in \rho^{\perp}$, and K_1 denote sectional curvature of the plane determined by X, Y and K_2 denote sectional curvature determined by $X,\ \rho$. Then

$$\begin{aligned} K_1 &= \frac{g(R(X,Y,Y),X)}{g(X,X)g(Y,Y) - \{g(X,Y)\}^2} \\ &= \frac{g[\frac{r}{2}g(Y,Y)X - g(X,Y)Y,\ \ X]}{g(X,X)g(Y,Y) - \{g(X,Y)\}^2} \\ &= \frac{r}{2} \end{aligned}$$

and

$$\begin{aligned} K_2 &= \frac{g(R(X,\rho,\rho),X)}{g(X,X)g(\rho,\rho) - \{g(X,\rho)\}^2} \\ &= \frac{g(0,X)}{-g(X,X)} \\ &= 0 \end{aligned}$$

Summing up we can state the following theorem:

Theorem 4. *A conformally flat perfect fluid $(WRS)_4$ spacetime obeying the Einstein equation without cosmological constant and having the basic vector field ρ as the velocity vector field has the following property:*

All planes perpendicular to ρ have sectional curvature $\frac{r}{2}$ and all planes containing ρ have sectional curvature 0.

By Karcher [12] a Lorentz manifold is called infinitesimally spatially isotropic relative to a timelike unit vector field ρ if its Riemannina curvature R satisfies the relations

$$R(X,Y)Z = l[g(Y,Z)X - g(X,Z)Y] \quad \text{for} \quad X,\ Y,\ Z \in \rho^{\perp}$$

and

$$R(X,\rho,\rho) = mX \quad \text{for} \quad X \in \rho^{\perp}$$

where $l,\ m$ are real valued functions on the manifold. By virtue of (23) and (24) we can state the following:

Theorem 5. *A conformally flat perfect fluid $(WRS)_4$ spacetime obeying the Einstein equation without cosmological constant and having the basic vector field $(WRS)_4$ as the velocity vector field of the fluid is infinitesimally spatially isotropic relative to the velocity vector field.*

Acknowledge. The authors are thankful to the referee for his valuable suggestions for the improvement of the paper. Also the second author is grateful to C.S.I.R. for providing a fellowship under Section No. F. No. 9/106(72)/2003–EMR I.

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(Received: June 23, 2004)
(Revised: August 19, 2004)

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O slabo Ricci simetričnim prostor-vrijeme mnogostrukostima

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Sadržaj

Cilj ovog rada je izučavanje slabih Ricci simetričnih prostor–vrijeme mnogostrukosti. Izmedju ostalog, dokazuje se da ako je u slabo Ricci simetričnom prostor–vremenu nenulte konstantne skalarne zakrivljenosti distribucija materije perfektno fluidna, onda su vektor ubrzanja i skalar ekspanzije jednaki nuli, i takav prostor–vrijeme ne dopušta toplotni fluks. Na kraju se izučava slabo Ricci simetrična prostor– vrijeme mnogostrukost konformno ravnog perfektnog fluida.