On contractions in probabilistic metric spaces

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Abstract. Two types of contractions are used for mappings defined on probabilistic metric spaces. The first type was introduced by V.M. Sehgal [15-16], the second type by T.L. Hicks [7]. Since then, many fixed point results were obtained. In this paper we introduce the concept of a probabilistic $g$-contraction, which is a generalization of a probabilistic contraction of Hicks’ type and prove some fixed point theorems.

1. Introduction

Throughout this paper $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and $I = [0, 1]$. A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non decreasing and left-continuous with $\inf F = 0$ and $\sup F = 1$.

In what follows we always denote by $D$ the set of all distribution functions. $D^+ = \{F : F \in D, F(0) = 0\}$ is the set of all distribution functions associated to non-negative, one-dimensional random variables.

For every $a \in \mathbb{R}^+$, a specific distribution function is defined by $H_a(t) = 0$ if $t \leq a$ and $H_a(t) = 1$ if $t > a$.

A mapping $T : I \times I \to I$ is called a t-norm if it satisfies the following conditions:

\begin{enumerate}
  \item[(T1)] $T(a, 1) = a$,
  \item[(T2)] $T(a, b) = T(b, a)$,
  \item[(T3)] $T(c, d) \geq T(a, b)$ if $c \geq d$ and $d \geq b$,
  \item[(T4)] $T(T(a, b), c) = T(a, T(b, c))$.
\end{enumerate}

The most used t-norms in probabilistic metric spaces theory are $T = \text{Min}$, $T = \text{Prod}$ and $T = T_m$, where $\text{Min}(a, b) = \min\{a, b\}$, $\text{Prod}(a, b) = a \cdot b$ and $T_m(a, b) = \max\{a + b - 1, 0\}$.

**Definition 1.** A probabilistic metric space of Menger type (briefly a Menger space) is an ordered triple $(S, F, T)$, where $S$ is a nonempty set, $F$ is

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a mapping from \( S \times S \) into \( D^+ \), \( T \) is a \( t \)-norm and the following conditions are satisfied:

\[
\begin{align*}
(M_1) \quad F_{x,y}(t) &= H_0(t) \text{ if and only if } x = y, \\
(M_2) \quad F_{x,y}(t) &= f_{y,x}(t), \text{ for all } t \in \mathbb{R}, \\
(M_3) \quad F_{x,z}(t_1 + t_2) &\geq T(F_{x,y}(t_1), F_{y,z}(t_2)), \text{ for all } x, y, z \in S \text{ and } t_1, t_2 \geq 0.
\end{align*}
\]

\( F(x, y) \) is denoted by \( F_{x,y} \), \( \mathcal{F} \) is called a probabilistic metric and \( (M_3) \) is a probabilistic version of the triangle inequality.

The study of probabilistic metric spaces was introduced by K. Menger in [10]. Since then, a lot of results were obtained in this area [1], [6], [13]. An important class of probabilistic metric spaces are the so-called random normed spaces. Let \( L \) be a linear space and let \( \mathcal{F} \) be a mapping on \( L \) with values in \( D^+ \). \( (F(x) \) is denoted by \( F_x \).

**Definition 2.** The ordered triple \((L, \mathcal{F}, T)\) is called a random normed space if the following conditions are satisfied:

\[
\begin{align*}
(N_1) \quad F_x &= H_0, \text{ if only if } x = \theta \text{ the null vector}, \\
(N_2) \quad F_{ax}(t) &= F_x\left(\frac{t}{|a|}\right), \text{ for } t \in \mathbb{R}, \alpha \in \mathbb{K}, \text{ where } \mathbb{K} \text{ is the field of scalars}, \\
(N_3) \quad F_{x+y}(t_1 + t_2) &\geq T(F_{x}(t_1), F_{y}(t_2)), \text{ for all } x, y \in L \text{ and } t_1, t_2 \in \mathbb{R}^+.
\end{align*}
\]

This notion was firstly studied in [17], [11]. For more details we refer [1], [13].

A uniformity on a Menger space \((S, \mathcal{F}, T)\) is defined by the family:

\[ \mathcal{V} = \{ V(\epsilon, \lambda) = \{ (x, y) \in S \times S : F_{u,v}(\epsilon) > 1 - \lambda, \epsilon > 0, \lambda \in (0, 1) \} \}. \]

In the sequel we will consider a Menger space \((S, \mathcal{F}, T)\) under a \( t \)-norm \( T \), which satisfies the weakest condition, \( \sup\{ T(t, t) : t < 1 \} = 1 \), that ensures the existence of a uniformity on \( S \).

### 2. On probabilistic \( g \)-contraction mappings

**Theorem 1.** Let \( g \) be an injective mapping defined on a Menger space \((S, \mathcal{F}, T)\) into itself. Then the following statements are true:

(a) The mapping \( \mathcal{F}^g \) defined on \( S \times S \) with values in \( D^+ \), by \( \mathcal{F}^g(x, y) = F_{g(x); g(y)} \) is a probabilistic metric on \( S \), that is, \((S, \mathcal{F}^g, T)\) is a Menger space under the same \( t \)-norm \( T \).

(b) If \( S_1 = g(S) \) and \((S_1, \mathcal{F}, T)\) is a complete Menger space then \((S, \mathcal{F}^g, T)\) is also a complete Menger space.

(c) If \((S_1, \mathcal{F}, T)\) is compact then \((S, \mathcal{F}^g, T)\) is also compact.

**Proof.** We will prove only the statement (c). Let \((x_n)_{n \geq 1}\) be a sequence in \( S \). Then \((u_n)_{n \geq 1}\) with \( u_n = g(x_n) \) is a sequence in \( S_1 \), which is a compact Menger space. Now, we can find a subsequence \( \{v_n : n \geq 1\} \subset \{u_n : n \geq 1\} \)
convergent to an element $v \in S_1$. This is equivalent to $F_{v,v}(t) \rightarrow H_0(t)$, $(n \rightarrow \infty)$, for every $t > 0$. If we set $y_n = g^{-1}(v_n)$ and $y = g^{-1}(v)$ then we have

$$F_{y_n,y}(t) = F_{g(y_n),g(y)}(t) = F_{v,v}(t) \rightarrow H_0(t), (n \rightarrow \infty),$$

for every $t > 0$. This show us that $(S,F^9,T)$ is a compact Menger space.

**Definition 3.** Let $f,g$ be two mappings defined on a Menger space $(S,F,T)$ with values into itself and let us suppose that $g$ is bijective. The mapping $f$ is called a probabilistic $g$–contraction with a constant $k \in (0,1)$ if

$$t > 0 \quad \text{and} \quad F_{g(x),g(y)}(t) > 1 - t \quad \text{implies} \quad F_{f(x),f(y)}(kt) > 1 - kt. \quad (C)$$

The notion of $g$–contraction is justified because the images of two points $x,y$ under the function $f$ are nearer than images of the same points under the function $g$.

**Theorem 2.** If $f$ is a probabilistic $g$–contraction, then we have :

(a) $g^{-1} \circ f$ is a continuous mapping on $(S,F^9,T)$ with values in $(S,F,T)$.

(b) $g^{-1} \circ f$ is a continuous mapping on $(S,F^9,T)$ with values into itself.

**Proof.** Let $(x_n)_{n \geq 1}$ be a sequence in $S$ such that $x_n \rightarrow x \in S$, under the probabilistic metric $F^9$. This implies that $F_{x_n,x}(t) \rightarrow H_0(t), (n \rightarrow \infty)$, for every $t > 0$. From the $g$–contraction condition $(C)$ it follows that $F_{f(x_n),f(x)}(t) \rightarrow H_0(t), (n \rightarrow \infty)$, for every $t > 0$. This show us that the mapping $f$ is continuous.

We observe that the above convergence implies $F_{g^{-1} \circ f, g^{-1} \circ f}(t) \rightarrow H_0(t), (n \rightarrow \infty)$, for every $t > 0$, that is, $F_{g^{-1} \circ f, g^{-1} \circ f}(t) \rightarrow H_0(t), (n \rightarrow \infty)$, for every $t > 0$. This shows us that the mapping $g^{-1} \circ f$ defined on the Menger space $(S,F^9,T)$ with values in itself is continuous.

**Remark 1.** The above concept of probabilistic $g$–contraction is a generalization of Hikcs’ probabilistic contraction [8] which can be obtained when $g$ is the identity on the Menger space $(S,F,T)$.

**Theorem 3.** If $f$ and $g$ are two mappings defined on a complete Menger space $(S,F,T)$ with values into itself, $g$ is bijective and $f$ is a $g$–contraction, there exists a unique point $p \in S$ such that $f(p) = g(p)$, ($p$ is considered a fixed point of the probabilistic $g$–contraction $f$). Moreover, $p = \lim_{n \rightarrow \infty} x_n$, where the sequence $(x_n)_{n \geq 1}$ is defined by recurrence relation $g(x_{n+1}) = f(x_n)$. 
Proof. If \( t > 1 \) then clearly \( F^g_{x,y}(t) > 1 - t \). By the definition of the probabilistic metric \( \mathcal{F}^g \) we have \( F_{x,y}(t) > 1 - t \). From the \( g \)-contraction condition (C) one obtains \( F_{x,y}(kt) > 1 - kt \). This means that \( F_{g^{-1}f_{x,y}}(kt) > 1 - kt \).

If we denote \( g^{-1} \circ f = h \), then by iterations it follows that

\[
F^g_{h^n x, h^n y}(k^n t) > 1 - k^n t.
\]

Since \( k \in (0,1) \), for every \( \epsilon > 0 \), \( \lambda \in (0,1) \) there exists a positive integer \( n(\epsilon, \lambda) \) such that \( k^n t \leq \min(\epsilon, \lambda) \), for every \( n \geq n(\epsilon, \lambda) \). Now, if we take into account that every distribution function is non decreasing we have

\[
F^g_{h^n x, h^n y}(\epsilon) \geq F^g_{h^n x, h^n y}(k^n t) > 1 - k^n t > 1 - \lambda.
\]

Let \( x_0 \) be fixed in \( X \) and let \( (x_n)_{n \geq 1} \) be the sequence of successive approximations defined by \( x_{n+1} = h(x_n) \), or equivalently, by \( g(x_{n+1}) = f(x_n) \). If we take \( x = x_m \) and \( y = x_0 \) then, from the above inequalities one obtains

\[
F^g_{x_{m+1}, x_0}(\epsilon) = F^g_{h^n x_m, h^n x_0}(\epsilon) > 1 - \lambda,
\]

for every \( n \geq n(\epsilon, \lambda) \) and \( m \geq 1 \). Therefore \( (x_n)_{n \geq 1} \) is a Cauchy sequence. Since \( (S, \mathcal{F}, T) \) is complete, then \( (S, \mathcal{F}^g, T) \) is complete and there exists a point \( p \in S \) such that the sequence \( (x_n)_{n \geq 1} \) converges, under the probabilistic metric \( \mathcal{F}^g \), to the point \( p \). As the mapping \( h \) defined on the Menger space \( (S, \mathcal{F}^g, T) \) with values in itself is continuous, it follows that \( h(p) = p \), that is, \( g^{-1} \circ f(p) = p \) or equivalently, \( f(p) = g(p) \). The uniqueness of the fixed point \( p \) also follows by the \( g \)-contraction condition (C).

Remark 2. When \( g \) is the identity on \( (S, \mathcal{F}, T) \) we obtain the known results from [7].

If we take in account that every metric space \( (S,d) \) can be made into a Menger space \( (S, \mathcal{F}, T) \), in a natural way, by setting \( F_{x,y}(t) = H_0(t - d(x,y)) \), for every \( x, y \in S \), \( t \in \mathbb{R}^+ \) and \( T = Min \) then, by Theorem 3 one obtains the following fixed point theorem for mappings defined on metric spaces.

Theorem 4. If \( f \) and \( g \) are two mappings defined on a complete metric space \( (S,d) \) with values into itself, \( g \) is bijective and \( f \) is a \( g \)-contraction, that is, there exists a constant \( k \in (0,1) \) such that

\[
d(f(x), f(y)) \leq kd(g(x), g(y)),
\]

for every \( x, y \in S \), then there exists a unique point \( p \in S \) such that \( f(p) = g(p) \).
Proof. We can suppose that \( d(x, y) \in [0, 1) \). If this is not true, we define the mapping \( d_1(x, y) = 1 - e^{-d(x, y)} \), then the pair \((S, d_1)\) is also a metric space and the uniformities defined by the metrics \( d \) and \( d_1 \) are equivalent.

Now, let us suppose that \( f \) is a \( g \)-contraction on \((S, d)\) and \( t > 0 \), \( F_{gx, gy}(t > 1 - t) \). Then we have \( H_0(t - d(gx, gy)) > 1 - t \). This implies \( d(gx, gy) < t \). So, we have \( d(fx, fy) < kt \), which implies \( H_0(kt - d(fx, fy)) = 1 > 1 - kt \). So, the mapping \( f \) is a \( g \)-contraction defined on \((S, F, T)\) with values into itself and the conclusion follows by the Theorem 3.

REFERENCES

O kontrakcijama na vjerovatnostnim metričkim prostorima

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Sadržaj

U radu se uvodi koncept vjerovatnostne $g$-kontrakcije, koji predstavlja generalizaciju vjerovatnostne kontrakcije tipa Hick i dokazuju se neki teoremi fiksne tačke.