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On contractions in probabilistic metric spaces

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Abstract. Two types of contractions are used for mappings defined on probabilistic metric spaces. The first type was introduced by V.M. Sehgal [15-16], the second type by T.L. Hicks [7]. Since then, many fixed point results were obtained. In this paper we introduce the concept of a probabilistic $g-$contraction, which is a generalization of a probabilistic contraction of Hicks' type and prove some fixed point theorems.

1. Introduction

Throughout this paper $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{I} = [0, 1]$. A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non decreasing and left-continuous with $\inf F = 0$ and $\sup F = 1$.

In what follows we always denote by $\mathcal{D}$ the set of all distribution functions. $\mathcal{D}^+ = \{F : F \in \mathcal{D}, F(0) = 0\}$ is the set of all distribution functions associated to non–negative, one–dimensional random variables.

For every $a \in \mathbb{R}^+$, a specific distribution function is defined by $H_a(t) = 0$ if $t \leq a$ and $H_a(t) = 1$ if $t > a$.

A mapping $T : I \times I \to I$ is called a t-norm if it satisfies the following conditions:

$(T1)$ $T(a, 1) = a$,
$(T2)$ $T(a, b) = T(b, a)$,
$(T3)$ $T(c, d) \geq T(a, b)$ if $c \geq d$ and $d \geq b$,
$(T4)$ $T(T(a, b), c) = T(a, (T(b, c))$.

The most used $t-$norms in probabilistic metric spaces theory are $T = Min$, $T = Prod$ and $T = T_m$, where $Min(a, b) = \min\{a, b\}$, $Prod(a, b) = a \cdot b$ and $T_m(a, b) = \max\{a + b - 1, 0\}$.

Definition 1. *A probabilistic metric space of Menger type (briefly a Menger space) is an ordered triple* $(S, \mathcal{F}, T)$*, where* S *is a nonempty set,* $\mathcal{F}$ *is*

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a mapping from $S \times S$ *into* $\mathcal{D}^+$, T *is a* $t-$*norm and the following conditions are satisfied :*

(M_1) $F_{x,y}(t) = H_0(t)$ *if and only if* $x = y$.

(M_2) $F_{x,y}(t) = F_{y,x}(t)$, *for all* $t \in \mathbb{R}$.

(M_3) $F_{x,z}(t_1 + t_2) \geq T(F_{x,y}(t_1), F_{y,z}(t_2))$, *for all* $x, y, z \in S$ *and* $t_1, t_2 \geq 0$.

$F(x, y)$ is denoted by $F_{x,y}$, $\mathcal{F}$ is called a probabilistic metric and (M_3) is a probabilistic version of the triangle inequality.

The study of probabilistic metric spaces was introduced by K. Menger in [10]. Since then, a lot of results were obtained in this area [1], [6], [13].

An important class of probabilistic metric spaces are the so called random normed spaces. Let L be a linear space and let $\mathcal{F}$ be a mapping on L with values in $\mathcal{D}^+$. ($F(x)$ is denoted by F_x.)

Definition 2. *The ordered triple* $(L, \mathcal{F}, T)$ *is called a random normed space if the following conditions are satisfied :*

(N_1) $F_x = H_0$, *if only if* $x = \theta$ *the null vector.*

(N_2) $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$, *for* $t \in \mathbb{R}$, $\alpha \in \mathbb{K}$, *where* $\mathbb{K}$ *is the field of scalars.*

(N_3) $F_{x+y}(t_1 + t_2) \geq T(F_x(t_1), F_y(t_2))$, *for all* $x, y \in L$ *and* $t_1, t_2 \in \mathbb{R}^+$.

This notion was firstly studied in [17], [11]. For more details we refer [1], [13].

A uniformity on a Menger space $(S, \mathcal{F}, T)$ is defined by the family :

$$\mathcal{V} = \{V(\varepsilon, \lambda) = \{(x, y) \in S \times S : F_{u,v}(\epsilon) > 1 - \lambda\}, \epsilon > 0, \lambda \in (0, 1)\}.$$

In the sequel we will consider a Menger space $(S, \mathcal{F}, T)$ under a $t-$norm T, which satisfies the weakest condition, $\sup\{T(t, t) : t < 1\} = 1$, that ensures the existence of a uniformity on S.

2. On probabilistic $g-$contraction mappings

Theorem 1. *Let* g *be a injective mapping defined on a Menger space* $(S, \mathcal{F}, T)$ *into itself. Then the following statements are true :*

(a) *The mapping* $\mathcal{F}^g$ *defined on* $S \times S$ *with values in* $\mathcal{D}^+$, *by* $\mathcal{F}^g(x, y) = F_{g(x),g(y)}$ *is a probabilistic metric on* S, *that is,* $(S, \mathcal{F}^g, T)$ *is a Menger space under the same* $t-$*norm* T.

(b) *If* $S_1 = g(S)$ *and* $(S_1, \mathcal{F}, T)$ *is a complete Menger space then* $(S, \mathcal{F}^g, T)$ *is also a complete Menger space.*

(c) *If* $(S_1, \mathcal{F}, T)$ *is compact then* $(S, \mathcal{F}^g, T)$ *is also compact.*

Proof. We will prove only the statement (c). Let $(x_n)_{n \geq 1}$ be a sequence in S. Then $(u_n)_{n \geq 1}$ with $u_n = g(x_n)$ is a sequence in S_1, which is a compact Menger space. Now, we can find a subsequence $\{v_n : n \geq 1\} \subset \{u_n : n \geq 1\}$

convergent to an element $v \in S_1$. This is equivalent to $F_{v_n,v}(t) \to H_0(t)$, $(n \to \infty)$, for every $t > 0$. If we set $y_n = g^{-1}(v_n)$ and $y = g^{-1}(v)$ then we have

$$F^g_{y_n,y}(t) = F_{g(y_n),g(y)}(t) = F_{v_n,v}(t) \to H_0(t), (n \to \infty),$$

for every $t > 0$. This show us that $(S, \mathcal{F}^g, T)$ is a compact Menger space.

Definition 3. *Let f, g be two mappings defined on a Menger space $(S, \mathcal{F}, T)$ with values into itself and let us suppose that g is bijective. The mapping f is called a probabilistic $g-$contraction with a constant $k \in (0,1)$ if*

$$t > 0 \quad \text{and} \quad F_{g(x),g(y)}(t) > 1 - t \quad \text{implies} \quad F_{f(x),f(y)}(kt) > 1 - kt. \qquad (C)$$

The notion of $g-$contraction is justified because the images of two points x, y under the function f are nearer than images of the same points under the function g.

Theorem 2. *If f is a probabilistic $g-$contraction, then we have :*
(a) *f is a continuous mapping on $(S, \mathcal{F}^g, T)$ with values in $(S, \mathcal{F}, T)$.*
(b) *$g^{-1} \circ f$ is a continuous mapping on $(S, \mathcal{F}^g, T)$ with values into itself.*

Proof. Let $(x_n)_{n \geq 1}$ be a sequence in S such that $x_n \to x \in S$, under the probabilistic metric $\mathcal{F}^g$. This implies that $F^g_{x_n,x}(t) \to H_0(t)$,$(n \to \infty)$, for every $t > 0$. From the $g-$contraction condition (C) it follows that $F_{f(x_n),f(x)}(t) \to H_0(t)$, $(n \to \infty)$, for every $t > 0$. This show us that the mapping f is continuous.

We observe that the above convergence implies $F_{gg^{-1}fx_n,gg^{-1}fx}(t) \to H_0(t)$, $(n \to \infty)$, for every $t > 0$, that is, $F^g_{g^{-1}fx_n,g^{-1}fx}(t) \to H_0(t)$, $(n \to \infty)$, for every $t > 0$. This shows us that the mapping $g^{-1} \circ f$ defined on the Menger space $(S, \mathcal{F}^g, T)$ with values in itself is continuous.

Remark 1. The above concept of probabilistic $g-$contraction is a generalization of Hikcs' probabilistic contraction [8] which can be obtained when g is the identity on the Menger space $(S, \mathcal{F}, T)$.

Theorem 3. *If f and g are two mappings defined on a complete Menger space $(S, \mathcal{F}, T)$ with values into itself, g is bijective and f is a $g-$contraction, there exists a unique point $p \in S$ such that $f(p) = g(p)$, (p is considered a fixed point of the probabilistic $g-$contraction f). Moreover, $p = \lim\limits_{n\to\infty} x_n$, where the sequence $(x_n)_{n \geqslant 1}$ is defined by recurrence relation $g(x_{n+1}) = f(x_n)$.*

Proof. If $t > 1$ then clearly $F^g_{x,y}(t) > 1-t$. By the definition of the probabilistic metric $\mathcal{F}^g$ we have $F_{gx,gy}(t) > 1-t$. From the $g-$contraction condition (C) one obtains $F_{fx,fy}(kt) > 1-kt$. This means that $F_{gg^{-1}fx,gg^{-1}fy}(kt) > 1-kt$. or equivalently $F^g_{g^{-1}fx,g-1fy}(kt) > 1-kt$.

If we denote $g^{-1} \circ f = h$, then by iterations it follows that

$$F^g_{h^n x, h^n y}(k^n t) > 1 - k^n t.$$

Since $k \in (0,1)$, for every $\epsilon > 0$, $\lambda \in (0,1)$ there exists a positive integer $n(\epsilon,\lambda)$ such that $k^n t \leqslant \min(\epsilon,\lambda)$, for every $n \geqslant n(\epsilon,\lambda)$. Now, if we take into account that every distribution function is non decreasing we have

$$F^g_{h^n x, h^n y}(\epsilon) \geqslant F^g_{h^n x, h^n y}(k^n t) > 1 - k^n t > 1 - \lambda.$$

Let x_0 be fixed in X and let $(x_n)_{n\geqslant 1}$ be the sequence of successive approximations defined by $x_{n+1} = h(x_n)$, or equivalently, by $g(x_{n+1}) = f(x_n)$. If we take $x = x_m$ and $y = x_0$ then, from the above inequalities one obtains

$$F^g_{x_{n+m}, x_n}(\epsilon) = F^g_{h^n x_m, h^n x_0}(\epsilon) > 1 - \lambda,$$

for every $n \geqslant n(\epsilon,\lambda)$ and $m \geqslant 1$. Therefore $(x_n)_{n\geqslant 1}$ is a Cauchy sequence. Since $(S,\mathcal{F},T)$ is complete, then $(S,\mathcal{F}^g,T)$ is complete and there exists a point $p \in S$ such that the sequence $(x_n)_{n\geqslant 1}$ converges, under the probabilistic metric $\mathcal{F}^g$, to the point p. As the mapping h defined on the Menger space $(S,\mathcal{F}^g,T)$ with values in itself is continuous, it follows that $h(p) = p$, that is, $g^{-1} \circ f(p) = p$ or equivalently, $f(p) = g(p)$. The uniqueness of the fixed point p also follows by the $g-$contraction condition (C).

Remark 2. When g is the identity on $(S,\mathcal{F},T)$ we obtain the known results from [7].

If we take in account that every metric space (S,d) can be made into a Menger space $(S,\mathcal{F},T)$, in a natural way, by setting $F_{x,y}(t) = H_0(t - d(x,y)))$, for every $x, y \in S$, $t \in \mathbb{R}_+$ and $T = Min$ then, by Theorem 3 one obtains the following fixed point theorem for mappings defined on metric spaces.

Theorem 4. *If f and g are two mappings defined on a complete metric space (S,d) with values into itself, g is bijective and f is a $g-$contraction, that is, there exists a constant $k \in (0,1)$ such that*

$$d(f(x),f(y)) \leqslant kd(g(x),g(y)),$$

for every $x, y \in S$, then there exists a unique point $p \in S$ such that $f(p) = g(p)$.

Proof. We can suppose that $d(x,y) \in [0,1)$. If this is not true, we define the mapping $d_1(x,y) = 1 - e^{-d(x,y)}$, then the pair (S, d_1) is also a metric space and the uniformities defined by the metrics d and d_1 are equivalent.

Now, let us suppose that f is a $g-$contraction on (S,d) and $t > 0$, $F_{gx,gy}(t > 1-t$. Then we have $H_0(t - d(gx,gy)) > 1-t$. This implies $d(gx,gy) < t$. So, we have $d(fx,fy) < kt$, which implies $H_0(kt - d(fx,fy)) = 1 > 1 - kt$. So, the mapping f is a $g-$contraction defined on $(S, \mathcal{F}, T)$ with values into itself and the conclusion follows by the Theorem 3.

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O kontrakcijama na vjerovatnostnim metričkim prostorima

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Sadržaj

U radu se uvodi koncept vjerovatnostne $g-$kontrakcije, koji predstavlja generalizaciju vjerovatnostne kontrakcije tipa Hick i dokazuju se neki teoremi fiksne tačke.