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Oscillatory behavior of solutions of three-dimensional delay difference systems

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Abstract. The authors study the oscillatory behavior of solutions of third order delay difference system of the form

$$\begin{aligned} \Delta x_n &= a_n y_{n-k}^{\alpha} \\ \Delta y_n &= b_n z_{n-\ell}^{\beta} \\ \Delta z_n &= -c_n x_{n-m}^{\gamma}, \end{aligned}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are real sequences, k, ℓ and m are nonnegative integers and α, β and γ are ratios of odd positive integers. Examples are provided to illustrate the results.

1. Introduction

In this paper, we are concerned with the delay difference system of the form

$$\begin{aligned} \Delta x_n &= a_n y_{n-k}^{\alpha} \\ \Delta y_n &= b_n z_{n-\ell}^{\beta} \\ \Delta z_n &= -c_n x_{n-m}^{\gamma}, \end{aligned} \tag{1}$$

where $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, n_0+2, \cdots\}$, n_0 is a nonnegative integer and Δ is the forward difference operator defined by $\Delta u_n = u_{n+1} - u_n$ subject to the following conditions:

(C_1) $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are nonnegative real sequences such that $\sum\limits_{n=n_0}^{\infty} a_n = \infty$, $\sum\limits_{n=n_0}^{\infty} b_n = \infty$, and $c_n \not\equiv 0$ for infinitely many values of n;

(C_2) k, ℓ and m are nonnegative integers and α, β and γ are ratio of odd positive integers.

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Let $\theta = \max\{k, \ell, m\}$. By a solution of the system (1), we mean a real sequence $\{(x_n, y_n, z_n)\}$ defined for all $n \geq n_0 - \theta$ that satisfies the system (1) for all $n \in \mathbb{N}(n_0)$. A solution $(\{x_n\}, \{y_n\}, \{z_n\})$ of the system (1) is nonoscillatory if each of its component is either eventually positive or eventually negative and oscillatory otherwise.

If $\{a_n\}$ and $\{b_n\}$ are positive then the system (1) can be reduced to a third order difference equation whose oscillatory behavior has been studied extensively in the literature. See for example [1, 2, 3, 9] and the references cited therein. However for the system (1), the oscillatory behavior is studied in [11] without delay arguments. All the results obtained in this paper state that " every solution $\{(x_n, y_n, z_n)\}$ of the system (1) is either oscillatory or $\lim_{n\to\infty} \inf |x_n| = 0$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = 0$."

The purpose of this paper is to obtain conditions under which all solutions of the system (1) are oscillatory. For related results corresponding to two-dimensional system one can refer to [4, 6, 7, 8, 10] and the references cited therein. Examples are provided to illustrate the relevance of the results discussed.

2. Some preliminary lemmas

In this section we state and prove some lemmas, which will be used in establishing our main results.

Lemma 2.1. *Let* $\{(x_n, y_n, z_n)\}$ *be a solution of the system* (1) *with* $\{x_n\}$ *nonoscillatory for* $n \in \mathbb{N}(n_0)$. *Then* $\{(x_n, y_n, z_n)\}$ *is nonoscillatory and* $\{x_n\}$,$\{y_n\}$, $\{z_n\}$ *are monotone for* $n \in \mathbb{N}(n_0)$.

Proof. Let $\{(x_n, y_n, z_n)\}$ be a solution of the system (1) with $\{x_n\}$ be nonoscillatory for $n \in \mathbb{N}(n_0)$. Then without loss of generality assume that $x_n > 0$ for $n \in \mathbb{N}(n_0)$ and hence from the third equation of the system (1) we have $\Delta z_n < 0$ for $n \geq N$. Thus $\{z_n\}$ is nonincreasing sequence for $n \geq N$ and therefore eventually of one sign for $n \geq N$. Since $\{a_n\}$ and $\{b_n\}$ have positive subsequences in view of condition (C_1), applying similar arguments to the second and the first equation of (1), we see that $\{y_n\}$ and $\{x_n\}$ are monotone for $n \geq N$. Hence $\{(x_n, y_n, z_n)\}$ is nonoscillatory and the proof is complete. □

Lemma 2.2. *Let* $\{(x_n, y_n, z_n)\}$ *be a nonoscillatory solution of the system* (1), *then there are only the following two cases for* $n \in \mathbb{N}(n_0)$ *sufficiently large:*

(I) $sgn\, x_n = sgn\, y_n = sgn\, z_n$,

(II) $sgn\, x_n = sgn\, z_n \neq sgn\, y_n$.

Proof. The proof is similar to that of Lemma 2.2 in [11] and hence the details are omitted. □

Lemma 2.3. [5] *If X and Y are nonnegative, then*

$$X^{\lambda} + (\lambda - 1)Y^{\lambda} - \lambda X Y^{\lambda-1} \geq 0, \quad \lambda > 1$$

where equality holds if and only if $X = Y$.

3. Oscillation results

In this section we establish conditions for the oscillation of all solutions of the system (1). We begin with the following theorem.

Theorem 3.1. *Consider the difference system* (1)*, subject to the conditions*

$$\alpha = \beta = \gamma = 1, \tag{2}$$

$$\sum_{n=n_0}^{\infty} c_n = \infty, \tag{3}$$

and

$$\sum_{t=n}^{n+m} c_t \left[\sum_{s=n}^{t} a_s \left(\sum_{j=s}^{t} b_j \right) \right] > 1. \tag{4}$$

Then every solution $\{(x_n, y_n, z_n)\}$ of the system (1) *is oscillatory.*

Proof. Let $\{(x_n, y_n, z_n)\}$ be a nonoscillatory solution of the system (1). Then choose an integer $N \in \mathbb{N}(n_0)$ such that for all $n \geq N$, the solutions $\{(x_n, y_n, z_n)\}$ of system (1) satisfy either Case (I) or (II) of Lemma 2.2.

First assume that the solution $\{(x_n, y_n, z_n)\}$ satisfies Case (I) of Lemma 2.2 for $n \geq N$. Without loss of generality assume that $x_{n-m} > 0$ for $n \geq N$. Define

$$w_n = \frac{z_n}{x_{n-\ell}}, \quad n \geqslant N.$$

Then , for $n \geqslant N$, we have

$$\Delta w_n = \frac{\Delta z_n}{x_{n-\ell}} - \frac{z_{n+1} \Delta x_{n-\ell}}{x_{n-\ell} x_{n-\ell+1}} \leq -c_n.$$

Summing the inequality from N to $j \geq N$, we obtain

$$\sum_{n=N}^{j} c_n \leqslant w_N$$

which contradicts (3) as $j \to \infty$.

Case (II). Let $s \in \mathbb{N}(n_0)$ be fixed and summing the third equation of (1) from s to $n-1$, we obtain

$$z_n - z_s + \sum_{t=s}^{n-1} c_t x_{t-m} = 0,$$

or

$$-b_n z_{n-\ell} + b_n \sum_{t=n-\ell}^{\infty} c_t x_{t-m} \leqslant 0,$$

or

$$-\Delta y_n + b_n \sum_{t=n}^{\infty} c_t x_{t-m} \leqslant 0.$$

Summing the last inequality from s to n and rearranging , we obtain

$$y_n + \sum_{t=n}^{\infty} \left(\sum_{s=n}^{t} b_s \right) c_t x_{t-m} \leqslant 0,$$

or

$$y_{n-k} + \sum_{t=n-k}^{\infty} \left(\sum_{s=n}^{t} b_s \right) c_t x_{t-m} \leqslant 0$$

$$\Delta x_n + a_n \sum_{t=n}^{\infty} \left(\sum_{s=n}^{t} b_s \right) c_t x_{t-m} \leqslant 0.$$

A final summation of the last inequality yields

$$\sum_{t=n}^{\infty} \left[\sum_{s=n}^{t} a_s \left(\sum_{j=s}^{t} b_j \right) \right] c_t x_{t-m} \leqslant x_n,$$

or

$$\sum_{t=n}^{n+m} \left[\sum_{s=n}^{t} a_s \left(\sum_{j=s}^{t} b_j \right) \right] c_t x_{t-m} \leqslant x_n. \tag{5}$$

Since $\{x_n\}$ is decreasing, (5) yields,

$$\sum_{t=n}^{n+m} c_t \left[\sum_{s=n}^{t} a_s \left(\sum_{j=s}^{t} b_j \right) \right] \leqslant 1,$$

which contradicts (4). The proof is compete. □

Example 3.1. Consider the difference system

$$\begin{aligned} \Delta x_n &= 4 y_{n-k} \\ \Delta y_n &= \frac{1}{2} z_{n-\ell} \\ \Delta z_n &= -4 x_{n-m}, \quad n \geq 1 \end{aligned} \tag{6}$$

where k, ℓ and m are even positive integers. All conditions of Theorem 3.1 are satisfied and hence all solutions of the system (6) are oscillatory. In fact $\{(x_n, y_n, z_n)\} = \left\{(-1)^n, \frac{(-1)^{n+1}}{2}, 2(-1)^n\right\}$ is one such solution of the system (6).

Theorem 3.2. *Consider the difference system* (1) *subject to the conditions* (3),

$$\alpha = \beta = 1 \text{ and } 0 < \gamma < 1 \tag{7}$$

and

$$\lim_{n\to\infty} \sup \sum_{t=n}^{n+m} c_t \left[\sum_{s=n}^{t} a_s \left(\sum_{j=s}^{t} b_j\right)\right] = \infty. \tag{8}$$

Then every solution $\{(x_n, y_n, z_n)\}$ *of the system* (1) *is almost oscillatory.*

Proof. Let $\{(x_n, y_n, z_n)\}$ be a nonoscillatory solution of the system (1). Then proceeding as in the proof of Theorem 3.1, we choose $N \in \mathbb{N}(n_0)$ so that Lemma 2.2 holds for $n \geq N$. First we consider Case (I).

Case (I). Define

$$w_n = \frac{z_n}{x_{n-\ell}^{\gamma}}, \quad n \geqslant N.$$

Then

$$\Delta w_n = \frac{\Delta z_n}{x_{n-\ell}^{\gamma}} - \frac{z_{n+1}\Delta x_{n-\ell}^{\gamma}}{x_{n-\ell}^{\gamma} x_{n-\ell+1}^{\gamma}} \leq -c_n, \quad n \geqslant N.$$

Summing the last inequality from N to $j \geq N$, we obtain

$$\sum_{n=N}^{j} c_n \leqslant w_N$$

which contradicts (3) as $j \to \infty$.

Case (II). Proceeding as in the proof of Theorem 3.1, we obtain

$$\sum_{t=n}^{n+m} c_t \left[\sum_{s=n}^{t} a_s \left(\sum_{j=s}^{t} b_j\right)\right] x_{t-m}^{\gamma} \leqslant x_n. \tag{9}$$

Since $\{x_n\}$ is positive decreasing and γ is such that $0 < \gamma < 1$, we have from (9),

$$\sum_{t=n}^{n+m} c_t \left[\sum_{s=n}^{t} a_s \left(\sum_{j=s}^{t} b_j\right)\right] \leqslant x_n^{1-\gamma}.$$

Next taking the limit supremum in the last inequality we see that

$$\lim_{n\to\infty}\sup\sum_{t=n}^{n+m} c_t\left[\sum_{s=n}^{t} a_s\left(\sum_{j=s}^{t} b_j\right)\right] < \infty$$

which contradicts (8). $\square$

Example 3.2. Consider the difference system

$$\begin{aligned}\Delta x_n &= 2ny_{n-1}\\ \Delta y_n &= \frac{2n+3}{n+2}\,z_{n-2}\\ \Delta z_n &= -\frac{2n+7}{(n+3)(n+4)}\,x_{n-3}^{\frac{1}{3}}, \quad n \geqslant 3.\end{aligned} \tag{10}$$

All conditions of Theorem 3.2 are satisfied and hence all solutions of the system (10) are oscillatory. In fact $\{(x_n, y_n, z_n)\} = \left\{((-1)^n,), \frac{(-1)^n}{n+1}, \frac{(-1)^{n+1}}{n+3}\right\}$ is one such solution of the system (10).

Theorem 3.3. *Consider the difference system* (1) *subject to the conditions*

$$\sum_{n=n_0}^{\infty} c_n\left(\sum_{s=n_0}^{n-m-1} a_s\left(\sum_{t=n_0}^{s-k-1} b_t\right)^{\alpha}\right)^{\gamma} = \infty, \tag{11}$$

$$\lim_{n\to\infty}\sup\sum_{t=n}^{n+m} a_t\left(\sum_{s=n}^{t} b_s\left(\sum_{j=s}^{t} c_j\right)^{\beta}\right)^{\alpha} = \infty \tag{12}$$

and

$$\alpha\beta\gamma < 1. \tag{13}$$

Then every solution $\{(x_n, y_n, z_n)\}$ *of the system* (1) *is oscillatory.*

Proof. Let $\{(x_n, y_n, z_n)\}$ be a nonoscillatory solution of the system (1). Then proceeding as in the proof of Theorem 3.1, we choose $N \in \mathbb{N}(n_0)$ so that Lemma 2.2 holds for $n \geq N$. First we consider Case (I).

Case (I). Summing the second equation of the system (1) from N to $n-k-1$, we obtain

$$y_{n-k} - y_N = \sum_{s=N}^{n-k-1} b_s z_{s-\ell}^{\beta}, \quad n \geqslant N+k+1$$

$$y_{n-k} \geqslant \sum_{s=N}^{n-k-1} b_s z_{s-\ell}^{\beta}, \quad n \geqslant N_1 \geqslant N+k+1. \tag{14}$$

Using the monotoncity of $\{z_n\}$ in (14), we have

$$y_{n-k}^{\alpha} \geqslant z_{s-k-\ell}^{\alpha\beta} \left(\sum_{s=N}^{n-k-1} b_s \right)^{\alpha}, \quad n \geqslant N_1. \tag{15}$$

Summing the first equation of the system (1) from N_1 to $n-m-1$ and using (15), we obtain

$$x_{n-m} \geqslant \sum_{s=N_1}^{n-m-1} a_s z_{s-k-\ell}^{\alpha\beta} \left(\sum_{t=N}^{s-k-1} b_t \right)^{\alpha}, \quad n \geqslant N_1+m+1. \tag{16}$$

From (16) and the monotoncity of $\{z_n\}$, we have

$$x_{n-m} \geqslant z_{n-(k+\ell+m)}^{\alpha\beta} \sum_{s=N_1}^{n-m-1} a_s \left(\sum_{t=N}^{s-k-1} b_t \right)^{\alpha}, \quad n \geqslant N_2 \geqslant N_1+m+1,$$

or

$$x_{n-m}^{\gamma} \geqslant z_n^{\alpha\beta\gamma} \left(\sum_{s=N_1}^{n-m-1} a_s \left(\sum_{t=N}^{s-k-1} b_t \right)^{\alpha} \right)^{\gamma}, \quad n \geqslant N_2. \tag{17}$$

Multiply (17) by $\frac{c_n}{z_n^{\alpha\beta\gamma}}$, using the third equation of the system (1), and then summing from N_2 to $n-1$, we obtain

$$\sum_{s=N_2}^{n-1} \frac{-\Delta z_s}{z_s^{\alpha\beta\gamma}} \geqslant \sum_{s=N_2}^{n-1} a_s \left(\sum_{t=N_1}^{s-m-1} a_t \left(\sum_{j=N}^{t-k-1} b_j \right)^{\alpha} \right)^{\gamma}, \quad n \geqslant N_2. \tag{18}$$

For $z_{n+1} < u < z_n$, we have

$$\int_{z_{n+1}}^{z_n} \frac{du}{u^{\alpha\beta\gamma}} \geqslant -\frac{\Delta z_n}{z_n^{\alpha\beta\gamma}}, \quad n \geqslant N_2. \tag{19}$$

Combining (18) and (19), we obtain

$$\int_0^{z_{N_2}} \frac{du}{u^{\alpha\beta\gamma}} \geqslant \sum_{n=N_2}^{\infty} c_n \left(\sum_{s=N_1}^{n-m-1} a_s \left(\sum_{t=N}^{s-k-1} b_t \right)^{\alpha} \right)^{\gamma}$$

which is a contradiction in view of (11) and (13).

Case (II). Let $s-\ell \in \mathbb{N}(n_0)$ be fixed and summing the third equation of (1) from $s-\ell$ to $n-1$, we obtain

$$z_n - z_{s-\ell} + \sum_{j=s-\ell}^{n-1} c_j x_{j-m}^{\gamma} = 0,$$

or

$$\left(\sum_{j=n}^{\infty} c_j x_{j-m}^{\gamma}\right)^{\beta} \leqslant z_{n-\ell}^{\beta}.$$

Multiplying both sides of the last inequality by b_n and then using the second equation of (1) and then summing from $s-k \in \mathbb{N}(n_0)$ to $n-1$ and rearranging we obtain

$$\left(\sum_{t=n}^{\infty} b_t \left(\sum_{s=n}^{t} c_s\right)^{\beta} x_{t-m}^{\gamma\beta}\right)^{\alpha} \leqslant -y_{n-k}^{\alpha}.$$

Multiplying the above inequality by a_n and using the first equation of (1) and then summing, we obtain

$$\sum_{t=n}^{\infty} a_t \left(\sum_{s=n}^{t} b_s \left(\sum_{j=s}^{t} c_j\right)^{\beta}\right)^{\alpha} x_{t-m}^{\alpha\beta\gamma} \leqslant x_n$$

or

$$\sum_{t=n}^{n+m} a_t \left(\sum_{s=n}^{t} b_s \left(\sum_{j=s}^{t} c_j\right)^{\beta}\right)^{\alpha} x_{t-m}^{\alpha\beta\gamma} \leqslant x_n. \tag{20}$$

Since $\{x_n\}$ is decreasing and from (13) and (20), we have

$$\lim_{n\to\infty} \sup \sum_{t=n}^{n+m} a_t \left(\sum_{s=n}^{t} b_s \left(\sum_{j=s}^{t} c_j\right)^{\beta}\right)^{\alpha} < \infty$$

which contradicts (12). $\square$

Example 3.3. Consider the difference system

$$\begin{aligned}
\Delta x_n &= 2(n+1)^{\frac{1}{3}} y_{n-3}^{\frac{1}{3}} \\
\Delta y_n &= \frac{2n+3}{n+1}\, z_{n-2} \\
\Delta z_n &= -\frac{2n+7}{(n+3)(n+4)}\, x_{n-1}^{\frac{3}{5}}, \quad n \geq 3.
\end{aligned} \tag{21}$$

All conditions of Theorem 3.3 are satisfied and hence all solutions of the system (21) are oscillatory.

Theorem 3.4. *Consider the difference system* (1) *subject to the conditions*

$$\alpha\beta\gamma = 1 \tag{22}$$

and

$$\sum_{t=n}^{n+m} a_t \left(\sum_{s=n}^{t} b_s \left(\sum_{j=s}^{t} c_j \right)^{\beta} \right)^{\alpha} > 1. \tag{23}$$

If there exists a positive decreasing sequence $\{\phi_n\}$ *such that*

$$\limsup_{j\to\infty} \sum_{n=n_0}^{j} \left(c_n\phi_n - \frac{1}{(\gamma+1)^{\gamma}} \frac{(\Delta\phi_n)^{\gamma+1}}{(\eta_{n-j-1}\phi_n)^{\gamma}} \right) = \infty, \tag{24}$$

where

$$\eta_n = a_n \left(\sum_{s=n_0}^{n-1} b_s \right)^{\alpha} > 0, \ \textit{for all } n \in \mathbb{N}(n_0). \tag{25}$$

Then all solutions $\{(x_n, y_n, z_n)\}$ *of the system* (1) *are oscillatory.*

Proof. Let $\{(x_n, y_n, z_n)\}$ be a nonoscillatory solution of the system (1). Then proceeding as in the proof of Theorem 3.1, we choose $N \in \mathbb{N}(n_0)$ so that Lemma 2.2 holds for $n \geq N$. First we consider Case (I). Define

$$w_n = \frac{\phi_n z_n}{x^{\gamma}_{n-m-1}}, \quad n \geqslant N_1 \geqslant N + m + 1.$$

Then, for $n \geqslant N_1$, we have

$$\Delta w_n = -c_n\phi_n + \frac{\Delta\phi_n}{\phi_{n+1}} w_{n+1} - \frac{\phi_n z_n \Delta x^{\gamma}_{n-m-1}}{x^{\gamma}_{n-m} x^{\gamma}_{n-m-1}}. \tag{26}$$

Using the mean value theorem for the function $r(t) = t^{\gamma}$, we have

$$\Delta x^{\gamma}_{n-m-1} \geqslant \begin{cases} \gamma x^{\gamma-1}_{n-m-1} \Delta x_{n-m-1}, & \text{if } r \geqslant 1 \\ \gamma x^{\gamma-1}_{n-m} \Delta x_{n-m-1}, & \text{if } r < 1. \end{cases} \tag{27}$$

From (26), (27) and in view of the behavior of $\{x_n\}$ and $\{z_n\}$ we obtain

$$\Delta w_n = -c_n\phi_n + \frac{\Delta\phi_n}{\phi_{n+1}} w_{n+1} - \frac{\gamma\phi_n w_{n+1} \Delta x_{n-m-1}}{\phi_{n+1} x_{n-m}}, \quad n \geqslant N_1. \tag{28}$$

Summing the second equation of the system (1) from N_1 to $n-k-1$ and then using the nonincreasing character of $\{z_n\}$ we obtain

$$y_{n-k} \geqslant z_n^{\beta} \left(\sum_{s=N_1}^{n-k-1} b_s \right) z_n^{\frac{1}{\gamma}}, \quad n \geqslant N_2 \geqslant N_1. \tag{29}$$

Now from the first equation of (1), (29) and (20), we have

$$\Delta x_n \geqslant a_n \left(\sum_{s=N_1}^{n-k-1} b_s \right)^{\alpha} z_n^{\frac{1}{\gamma}}, \quad n \geqslant N_1$$

or

$$\Delta x_{n-m-1} \geqslant \eta_{n-m-1} z_{n-m-1}^{\frac{1}{\gamma}} \geqslant \eta_{n-m-1} z_{n+1}^{\frac{1}{\gamma}}, \quad n \geqslant N_1 \tag{30}$$

since $\{z_n\}$ is nonincreasing. Using (30) and (28) and simplifying we obtain

$$\Delta w_n \leqslant -c_n \phi_n + \frac{\Delta \phi_n}{\phi_{n+1}} w_{n+1} - \frac{\gamma \phi_n \eta_{n-m-1}}{\phi_{n+1}^{1+\frac{1}{\gamma}}} w_{n+1}^{1+\frac{1}{\gamma}}, \quad n \geqslant N_2 \geqslant N_1.$$

Set

$$X = (\gamma \phi_n \eta_{n-m-1})^{\frac{\gamma}{1+\gamma}} \frac{w_{n+1}}{\phi_{n+1}}, \quad \lambda = \frac{\gamma+1}{\gamma} > 1$$

and

$$Y = \left(\frac{\gamma}{\gamma+1} \right)^{\gamma} \left(\frac{\Delta \phi_n}{\phi_{n+1}} \right)^{\gamma} \left[\gamma^{-\left(\frac{\gamma}{\gamma+1}\right)} (\phi_n \eta_{n-m-1})^{-\frac{\gamma}{1+\gamma}} \phi_{n+1} \right]^{\gamma}$$

in Lemma 2.3, to conclude that

$$\frac{\Delta \phi_n}{\phi_{n+1}} w_{n+1} - \frac{\gamma \phi_n \eta_{n-m-1}}{\phi_{n+1}^{1+\frac{1}{\gamma}}} w_{n+1}^{1+\frac{1}{\gamma}} \leqslant \frac{1}{(\gamma+1)^{\gamma}} \frac{(\Delta \phi_n)^{\gamma+1}}{\eta_{n-m-1}^{\gamma} \phi_n^{\gamma}}$$

and therefore

$$\Delta w_n \leqslant -c_n \phi_n + \frac{1}{(\gamma+1)^{\gamma}} \frac{(\Delta \phi_n)^{\gamma+1}}{\eta_{n-m-1}^{\gamma} \phi_n^{\gamma}}, \quad n \geqslant N_2.$$

Summing both sides of the last inequality from N_2 to $j \geq N_1$, we obtain

$$w_{j+1} - w_{N_2} \leqslant - \sum_{n=N_2}^{j} \left[c_n \phi_n - \frac{1}{(\gamma+1)^{\gamma}} \frac{(\Delta \phi_n)^{\gamma+1}}{\eta_{n-m-1}^{\gamma} \phi_n^{\gamma}} \right] \to -\infty$$

as $j \to \infty$ which is a contradiction to the fact that $w_j > 0$ for $j \geq N_2$.

Case (II). Proceeding as in the proof of Theorem 3.3, we obtain (20). Now using the nonincreasing behavior of $\{x_n\}$ and condition (22), we obtain a contradiction to (23). $\square$

In the case of $\alpha\beta\gamma > 1$, we are unable to find the conditions under which all solutions of the system (1) are oscillatory. However, we establish the following result.

Theorem 3.5. *Consider the difference system* (1) *subject to the conditions*

$$\alpha\beta\gamma > 1, \tag{31}$$

$$\sum_{n=n_0}^{\infty} b_n \left(\sum_{s=n}^{\infty} c_s\right)^{\alpha} = \infty \tag{32}$$

and

$$\sum_{n=n_0}^{\infty} a_n \left(\sum_{s=n_0}^{n-k-1} b_s\right)^{\alpha} \left(\sum_{s=n+m+1}^{\infty} c_s\right)^{\beta} = \infty \tag{33}$$

hold. Then every solution $\{(x_n, y_n, z_n)\}$ *of the system* (1) *is either oscillatory or* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = 0$.

Proof. Let $\{(x_n, y_n, z_n)\}$ be a nonoscillatory solution of the system (1). Then proceeding as in the proof of Theorem 3.1, we see that $\{(x_n, y_n, z_n)\}$ satisfies one of the two cases in Lemma 2.2 for $n \geq N$. First consider Case (I). In this case, from the third equation of the system (1) and using the nondecreasing behavior of $\{x_n\}$, we have

$$z_n \geqslant x_{n-m}^{\gamma} \sum_{s=n}^{\infty} c_s, \qquad n \geqslant N. \tag{34}$$

Further, summing the second equation of the system (1) from N to $n-1$ and then using the nonincreasing character of $\{z_n\}$ we obtain

$$y_n \geqslant z_{n-\ell}^{\beta} \left(\sum_{s=N}^{n-1} b_s\right), \quad n \geqslant N$$

or

$$y_{n-k} \geqslant z_{n-k-\ell}^{\beta} \left(\sum_{s=N}^{n-k-1} b_s\right), \quad n \geqslant N_1 \geqslant N+k+1. \tag{35}$$

From (34), (35) and the first equation of system (1), we have

$$\Delta x_n \geqslant a_n \left(\sum_{s=N}^{n-k-1} b_s\right)^{\alpha} \left(\sum_{s=n+m+1}^{\infty} c_s\right)^{\beta} x_{n+1}^{\alpha\beta\gamma}$$

or

$$\sum_{s=N}^{n-1}\frac{\Delta x_s}{x_{s+1}^{\alpha\beta\gamma}} \geqslant \sum_{s=N_1}^{n-1} a_s\left(\sum_{t=N}^{s-k-1} b_t\right)^{\alpha}\left(\sum_{t=s+m+1}^{\infty} c_t\right)^{\beta}, \quad n\geqslant N_1. \tag{36}$$

For $x_n < u < x_{n+1}$, we have

$$\int_{x_n}^{x_{n+1}}\frac{du}{u^{\alpha\beta\gamma}} \geqslant \frac{\Delta x_n}{x_{n+1}^{\alpha\beta\gamma}}, \quad n\geqslant N_1. \tag{37}$$

Combining (36) and (37), we obtain

$$\int_{x_{N_1}}^{\infty}\frac{du}{u^{\alpha\beta\gamma}} \geqslant \sum_{n=N_1}^{\infty} a_n\left(\sum_{s=N}^{n-k-1} b_s\right)^{\alpha}\left(\sum_{s=n+m+1}^{\infty} c_s\right)^{\beta},$$

which is a contradiction in view of (31) and (33).

Case (II). Now from the first equation of (1), we see that $\{x_n\}$ is nonincreasing for $n \geq N$ and therefore $\lim\limits_{n\to\infty} x_n = L_1 < \infty$. Hence from Lemma 2.3 in [11], we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = 0.$$

We shall prove that $\lim\limits_{n\to\infty} x_n = 0$. Let $L_1 > 0$. Then there is an integer $N_1 > N+m$ such that $x_{n-m} > d_1 > 0$ for $m \geq N_1$. Now summing the third equation (1) from n to ∞ and then using $x_{n-m} > d_1$ for $m \geq N_1$, we obtain

$$z_n \geqslant d_1^{\gamma}\sum_{s=n}^{\infty} c_s, \quad n\geqslant N_1.$$

Suppose β is a ratio of odd positive integers and $\{z_n\}$ is nonincreasing, we have from the last inequality

$$z_{n-\ell}^{\beta} \geqslant d_1^{\gamma\beta}\left(\sum_{s=n}^{\infty} c_s\right)^{\beta}, \quad n\geqslant N_1. \tag{38}$$

Summing the second equation (1) from N_1 to $n-1$ and then using (38), we obtain

$$y_n \geqslant y_{N_1} + d_1^{\gamma\beta}\sum_{s=N_1}^{n-1} b_s\left(\sum_{t=s}^{\infty} c_t\right)^{\beta}, \quad n\geqslant N_1.$$

In view of (32), the last inequality implies for that $\lim_{n\to\infty} y_n = \infty$, which is a contradiction. Therefore $\lim_{n\to\infty} x_n = 0$. □

We conclude this paper with the following example.

Example 3.4. Consider the difference system

$$\begin{aligned}\Delta x_n &= (1+(-1)^n)y_{n-2}^3\\ \Delta y_n &= n z_{n-3}^{\frac{1}{3}}\\ \Delta z_n &= -\frac{1}{n(n+1)}x_{n-1}^3, \quad n \geqslant 3.\end{aligned} \tag{39}$$

All conditions of Theorem 3.5 are satisfied for the system (39) and hence every solution $\{(x_n, y_n, z_n)\}$ for the system (39) is either oscillatory or $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = 0$.

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Oscilatorno djelovanje rješenja trodimenzionalnih diferentnih sistema sa kašnjenjem

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Sadržaj

U radu se izučava oscilatorno djelovanje rješenja diferentnog sistema trećeg reda sa kašnjenjem, koji ima oblik

$$\begin{aligned}\Delta x_n &= a_n y_{n-k}^{\alpha}\\ \Delta y_n &= b_n z_{n-\ell}^{\beta}\\ \Delta z_n &= -c_n x_{n-m}^{\gamma},\end{aligned}$$

gdje su $\{a_n\}$, $\{b_n\}$ i $\{c_n\}$ realne sekvence, k, ℓ i m nenegativni cijeli brojevi i α, β, i γ omjeri neparnih pozitivnih cijelih brojeva. Dati su primjeri za ilustraciju rezultata.