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On commutativity of rings with multiplicative power endomorphism conditions on certain subsets

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Abstract. We prove the following theorem: Let $n \geq 1$. Let R be a ring with 1, and let $S = N(R) \cup J(R)$. If R has property T and $(xy)^k = x^k y^k$ for all $x, y \in R - S$ and $k = n, n+2, n+4$, then R is commutative.

S. Ligh and A. Richoux [8] proved that if a ring R with 1 satisfies $(xy)^k = x^k y^k$ for all $x, y \in R$ and positive integers $k = n, n+1, n+2$ then R is commutative. Ashraf and Quadri [1] proved that if R is a ring with 1 satisfying $(xy)^n = y^n x^n$ for all $x, y \in R - N(R)$ and fixed integer $n > 1$ and the commutators in R are $n(n+1)$ torsion free, then R is commutative. In this paper we prove the result stated in the abstract.

Throughout R will denote an associative ring with unity 1, $Z(R)$ the centre of R, $J(R)$ the Jacobson radical of R, $C(R)$ the commutator ideal of R, $N(R)$ the set of all nilpotent elements of R, $S = N(R) \cup J(R)$ and $[x, y] = xy - yx$.

Lemma l. ([5], p. 221). *If $x, y \in R$ and $[x, [x, y]] = 0$ then $[x^m, y] = mx^{m-1}[x, y]$ for all positive integers m.*

Lemma 2. ([9]). *Let R be a ring with 1 and let $f : R \to R$ be a function such that $f(x+1) = f(x)$ for all $x \in R$. If for some positive integer $n, x^n f(x) = 0$ for all $x \in R$ then necessarily $f(x) = 0$.*

Lemma 3. *Let R be a ring with 1 and let $f : R \to R$ be a function such that $f(x) = f(x+1)$ for all $x \in R$. If $f(x)(x+t)^m x^n = 0$ for all $x \in R$ and some fixed positive integers t, m and n, then $(t+l)^{mn} f(x) = 0$.*

Proof. Replace x by $(x+1)$ in $f(x)(x+t)^m x^n = 0$ and expanding $\big(x + (t+1)\big)^m$ and $(x+1)^n$ by the Binomial Theorem and multiplying by

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$(x+t)^m x^{n-1}$ from right, we get $(t+1)^m f(x)(x+t)^m x^{n-1} = 0$. By continuing this process, we get $(t+1)^{mn} f(x)\,(x+t)^m = 0$. Now replace x by $(x+1)$ in this equation and expanding the factor $\big((x+t)+1\big)^m$ by the Binomial Theorem and multiplying by $(x+t)^{m-1}$ from right, we get $(t+1)^{mn} f(x)(x+t)^{m-1} = 0$. Thus $(t+1)^{mn} f(x) = 0$.

Definition. *Property T : If $u^2 \in Z(R)$ for all invertible elements u of R then $N(R) \subseteq Z(R)$.*

Theorem. *Let $n \geq 1$. Let R be a ring with 1, and let $S = N(R) \cup J(R)$. If R has property T and*

$$(xy)^k = x^k y^k \text{ for all } x, y \in R - S \text{ and } k = n, n+2, n+4, \tag{$*$}$$

then R is commutative.

Lemma 4. *Let R be a ring with 1 and $n \geq 1$. Let u be an invertible element of R and $x \in R$ such that $(ux)^k = u^k x^k$, $k = n, n+2, n+4$. Then $[x^2, u^2]x^{n+2} = 0$.*

Proof. We have $u^{n+2}x^{n+2} = (ux)^{n+2} = (ux)^n (ux)^2 = u^n x^n (ux)^2$, so

$$u^2 x^{n+2} = x^n (ux)^2. \tag{1}$$

Similarly,

$$u^2 x^{n+4} = x^{n+2}(ux)^2. \tag{2}$$

Left–multiplying (1) by x^2 and comparing the result with (2) gives $[x^2, u^2]x^{n+2} = 0$.

Lemma 5. *Let R be a ring with 1. Suppose that $m[x^2, y] = 0$ for all $x \in R$ and that $(xy)^k = x^k y^k$ for all $x \in R$ and $k = n, n+2$. Then $m[x, y]y^{n+1} = 0$ for all $x \in R$.*

Proof. $x^{n+2}y^{n+2} = (xy)^{n+2} = (xy)^n (xy)^2 = x^n y^n (xy)^2$. Therefore $mx^{n+2}y^{n+2} = mx^n (xy)^2 y^n$ or $mx^{n+1}[x, y]y^{n+1} = 0$ for all $x \in R$; and by Lemma 2, $m[x, y]y^{n+1} = 0$ for all $x \in R$.

Lemma 6. *Let R be a ring with 1 with $N \subseteq Z$. If $(xy)^k = x^k y^k$ for all $x, y \in R$ and $k = n, n+2, n+4$, then R is commutative.*

Proof. By a well known theorem of Herstein, the commutator ideal of R is nil, hence commutators are central. Now the equality $(xy)^k x = x(yx)^k$ gives $x^k y^k x = xy^k x^k$ that is $x[x^{k-1}, y^k]x = 0$. Hence, since commutators are central, $(k-1)kx^ky^{k-1}[x,y]=0$ for all $x,y\in R$, $k=n,n+2,n+4$. Therefore by Lemma 2, we have $(k-1)k[x,y]=0$ for all $x,y\in R$, $k=n,n+2,n+4$.

Now $\big((n-1)n,(n+1)(n+2),(n+3)(n+4)\big)=2$ for $n>1$; and $(6,20)=2$. Thus, we have $2[x,y]=0$ for all $x,y\in R$. It follows that $[x^2,y]=2x[x,y]=0$ for all $x,y\in R$. By Lemmas 5 and 2 we conclude that R is commutative.

Proof of Theorem. Let u and v be invertible elements of R. By Lemma 4, $[u^2,v^2]=0$ and by (1) with $x=v$, we get $[u,v]=0$. Letting $a\in S$ and noting that $1+a$ is invertible, we see that $[u,a]=0$. This fact, together with (*), yields

$$(ux)^k=u^kx^k \ \text{ and } \ (xu)^k=x^ku^k \ \text{ for all } \ x\in R,\ k=n,n+2,n+4. \qquad (3)$$

It now follows by Lemma 4 that

$$[x^2,u^2]\,x^{n+2}=0 \ \text{ for all } \ x\in R \ \text{ and all invertible } \ u. \qquad (4)$$

Replacing x in turn by $x+1$ and $x+2$ and right multiplying each of the resulting equations by x^{x+2} yields

$$4[x,u^2](x+1)^{n+2}x^{n+2}=0=4\,[x,u^2](x+2)^{n+2}x^{n+2}.$$

By Lemma 3, we get $2^{(n+2)^2}4\,[x,u^2]=0$ and $3^{(n+2)^2}4\,[x,u^2]=0$, so that

$$4\,[x,u^2]=0 \ \text{ for all } \ x\in R \ \text{ and all invertible } \ u. \qquad (5)$$

If we now replace x by $x+1$ in (4) and right multiply the result by $2x^{n+1}$, we get $2\,[x^2,u^2]\,x^{n+1}=0$. By continuing the process, we ultimately arrive at $2\,[x^2,u^2]=0$ for all $x\in R$ and invertible u. It now follows from Lemma 5 that $2\,[x,u^2]=0$ for all $x\in R$ and invertible u. By replacing x by $x+1$ in (4) and right multiplying by x^{n+1}, we get $[x^2,u^2]x^{n+1}=0$ and after several repetitions, we have

$$[x^2,u^2]=0 \text{ for all } \ x\in R \text{ and invertible } \ u. \qquad (6)$$

An appeal to Lemma 5 now yields

$$[x,u^2]=0 \text{ for all } \ x\in R \text{ and invertible } \ u, \qquad (7)$$

and it follows by Property T that $N\subseteq Z$. Thus, N is an ideal, necessarily contained in $J(R)$; and $S=J(R)$. Our hypotheses on R now show that $R/J(R)$ satisfies the identities $(xy)^k=x^ky^k$ for $k=n,n+2,n+4$. Hence by Herstein's theorem $R/J(R)$ is commutative. Thus

$$[x,y]\in J(R) \ \text{ for all } \ x,y\in R. \qquad (8)$$

Since invertible elements of R commute, $J(R)$ is commutative ideal, hence $J(R)^2$ is central. By (8) we see that R satisfies the identity $\big[[x,y],[z,w]\big] = 0$. Hence by Theorem 1 of [2], $C(R)$ is nil and commutators are central.

Recalling (7) and using the fact that $J(R)^2 \subseteq Z$, we see that for all $x \in R$ and $y \in J(R)$, $\lfloor x,(1+y)^2 \rfloor = 0 = 2\,[x,y]$. It follows that $[x^2,y]=0$ for all $x \in R$ and $y \in J(R)$ and hence $[x^2, 1+y]=0$ for all $x \in R$ and $y \in J(R)$. By (3) and Lemma 5, we get $J(R) \subseteq Z$. It is now apparent that $(xy)^k = x^k y^k$ for all $x,y \in R$, $k = n, n+2, n+4$, so R is commutative by Lemma 6.

Corollary 1. *Let R be a ring with 1, $S = N(R) \cup J(R)$ and n be a positive integer. If $(xy)^k = y^k x^k$ for all $x,y \in R - S$ and $k = n, n+2, n+4$, then R is commutative.*

Proof. Clearly, $(xy)^{k+1} = x(yx)^k y = x x^k y^k y = x^{k+1} y^{k+1}$ for all $x,y \in R{-}S$ and $k = n, n+2, n+4$. Thus, we need only show that R has property T.

Suppose that $u^2 \in Z$ for all invertible u in R. By an argument similar to that by which we obtained (3), we get

$$(xu)^k = u^k x^k \text{ for all invertible } u \text{ and all } x \in R,\ k = n, n+2, n+4. \tag{9}$$

Thus $u^{n+2}x^{n+2} = (xu)^{n+2} = (xu)^n (xu)^2 = u^n x^n (xu)^2$, so that $u^2 x^{n+2} = x^n (xu)^2$, $x^{n+2}u^2 = x^n (xu)^2 = x^n (xux)u$ and $x^{n+2}u = x^n xux$. Therefore $x^{n+1}[x,u] = 0$ for all $x \in R$ and all invertible u. It follows from Lemma 2 that invertible elements are central and hence $N \subseteq Z$. So R has property T.

Corollary 2. *Let n be a positive integer and let R be a ring with 1 satisfying $(xy)^k = x^k y^k$ for all $x,y \in R - S$, $k = n, n+2, n+4$. Suppose that one of the following holds:*

(i) *n is even.*

(ii) *$2\,[x,a] = 0$ implies $[x,a] = 0$ for all $x \in R$ and $a \in N$.*

Then R is commutative.

Proof. Again we need to show that R has property T, so begin with the assumption that $u^2 \in Z$ for all invertible u. If (i) holds, (3) implies (9) and we can argue as in the proof of Corollary 1.

Assume now that (ii) holds. If $a \in N$ and $a^{2^t} \in Z$, then $[x,(1+a^{2^{t-1}})^2] = 0 = 2\,[x, a^{2^{t-1}}]$. Hence $[x, a^{2^{t-1}}] = 0$ and $a^{2^{t-1}} \in Z$. By backward induction $a \in Z$. Hence R has property T.

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O komutativnosti prstenova s uslovima multiplikativnog endomorfizma u nekim podskupovima

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Sadržaj

U radu se dokazuje slijedeći teorem: Neka je $n \geq 1$. Neka je R prsten sa 1−com, i neka je $S = N(R) \cup J(R)$. Ako R ima svojstvo T i $(xy)^k = x^k y^k$ za sve $x, y \in R - S$ i $k = n, n+2, n+4$, tada je R komutativan.