RADOVI MATEMATIČKI
Vol. 13 (2004), 23–32

Maximal divisible subgroups in modular group algebras of p-mixed and p-splitting Abelian groups

Peter V. Danchev (Bulgaria)

Abstract. Suppose FG is the group algebra of an Abelian group G over the field F with char$F = p \neq 0$. The main purpose and result of this paper is the calculation of the maximal divisible subgroup of the normed unit group $V(FG)$ in FG in the case when the torsion part tG of G is p-primary. As a corollary, it is shown that $V(FG)$ is reduced if and only if G is reduced, provided tG is p-torsion. Moreover, the structure of the maximal divisible subgroup of $V(FG)$ is discussed when G is p-splitting. This extends a result of N. Nachev [Na].

1. Introduction

We first introduce some notation. Let R be a commutative ring with identity of prime characteristic p with a unit group $U(R)$, with a nil-radical (a Baer radical) $N(R)$ and with a maximal (p-)divisible (perfect) subring R_d. Let F be a field of characteristic p with a maximal perfect subfield F_d (it is easy to see that $F_d = F^{p^\omega}$). For G an Abelian group, $tG = \coprod_p G_p$ (G_p are called the p-primary components in G) will denote its maximal torsion subgroup, and dG, respectively G^*, will denote its maximal divisible subgroup, respectively its maximal p-divisible subgroup. Further, for the notations and terminology to the Abelian group theory, we shall follow the monographs of L.Fuchs [F]. For instance, $r_0(G)$ and $r_q(G)$ designate the torsion-free rank and the q-rank(q is a prime) of G, respectively. RG denotes the R-group algebra of G with a group of all normalized units denoted by $V(RG)$. We let $S(RG) = V_p(RG)$ denote the Sylow p-subgroup of $V(RG)$, i.e. its p-component. As usual, we shall let $I(RG; H)$ denote the relative augmentation ideal of RG about the subgroup H of G.

2000 Mathematics Subject Classification: Primary: 16S34, 16U60, 20C07; Secondary: 20K10, 20K21.

Keywords and phrases: Modular group algebras, maximal divisible subgroups, p-mixed Abelian groups, p-splitting Abelian groups.

In the present study a convenient explicit formula for $dV(FG)$ is given assuming tG as a p-group. Moreover, as a consequence, the isomorphism class of $dV(FG)$ in this case is provided. In particular, a criterion is obtained $V(FG)$ to be reduced when tG contains only p-torsion. Besides, $dV(FG)$ is partially described provided G is p-splitting, i.e. when G_p is a direct factor of G. Finally, $d(S(RG)/G_p)$ and $d(V(FG)/G)$ are completely computed, where $tG = G_p$ in the second case.

Now, for the sake of completeness and for the convenience of the reader, we summarize below some needed and well-known results, namely:

Proposition. ([M]) *Let G be an Abelian group such that $tG = G_p$. Then $V(FG) = G(1 + I(FG; G_p))$.*

Remark. Owing to [DA] or [DAN] we have proved that $S(RG) = 1 + I(RG; G_p)$ when $N(R) = 0$. Thus the above formula of May can be replaced by $V(FG) = GS(FG)$.

Theorem. ([N] and [NA]).

$$dS(RG)=S(R_dG^*)\cong\dot{\sum_{\lambda}}\mathbf{Z}\,(p^\infty),$$

where

$$\lambda=\begin{cases}\max(|R_d|,|G^*|), & \textit{if } (G^*)_p=(G_p)^*\neq 1;\\ \max(|N(R_d)|,|G^*|), & \textit{if } (G^*)_p = 1,\\ G^*\neq 1 \textit{ and } N(R_d)\neq 0; & \\ 0, \textit{ in the remaining cases.} & \end{cases}$$

2. Main results

Here we can formulate the main statements which are the following:

I. *The characterization of $d\,[S(RG)/G_p]$ and $d\,[V(FG)/G]$.*

Proposition 1.

$$d\,[S(RG)/G_p] = S(R_dG^*)G_p/G_p \cong S(R_dG^*)/(G^*)_p. \tag{$*$}$$

$$\begin{aligned} d\,[V(FG)/G] &= V(F_dG^*)G/G \cong V(F_dG^*)/G^* \\ &\cong S(F_dG^*)G_p/G_p \cong S(F_dG^*)/(G^*)_p, \end{aligned} \tag{$**$}$$

when $tG = G_p$.

Corollary 2. *$S(RG)$ is reduced if and only if $N(R_d)=0$ and G_p is reduced; or $N(R_d)\neq 0$ and G is p-reduced. Also $S(RG)/G_p$ is reduced if and only if $N(R_d)=0$, G is not p-reduced and G_p is reduced; or $G^*\neq 1$ is a p-group of two elements and R_d is a field of two elements; or G is p-reduced. Moreover assuming $tG=G_p$, $V(FG)/G$ is reduced if and only if G is not p-reduced and G_p is reduced; or G^* and F_d have cardinalities 2; or G is p-reduced.*

II. *The construction of $dV(FG)$ for $tG=G_p$.*

Theorem 3. *Let τ be any ordinal and G be an Abelian group whose tG is p-torsion. Then the following explicit formula is valid*

$$V^{\tau}(FG)=G^{\tau}S(F^{p^{\omega}}G^{p^{\omega\tau}}).$$

The following statement appeared in [DANCHEV].

Corollary 4. *Suppose G is an Abelian group for which tG is p–primary. Then $dV(FG)=dGdS(FG)=dGS(F_dG^*)$. Thus $dV(FG)/dS(FG)\cong dG/d(G_p)$.*

Corollary 5. *Suppose G is Abelian so that tG is a p-group. Then the following isomorphisms hold:*

$$dV(FG)\cong dG\times[S(F_dG^*)/(G^*)_p]\cong dG\times[S(F_dG^*)G_p/G_p],$$

where $(G^*)_p=(dG)_p=d(G_p)$.

Corollary 6. *Suppose G is Abelian such that tG is p-primary. Then $V(FG)$ is reduced if and only if G is reduced.*

3. Proofs of the main results

We will first prove some preliminary claims.

Lemma 7. *For each ordinal τ the following are true $U^{p^{\tau}}(R)=U(R^{p^{\tau}})$, $V^{p^{\tau}}(RG)=V(R^{p^{\tau}}G^{p^{\tau}})$ and $S^{p^{\tau}}(RG)=S(R^{p^{\tau}}G^{p^{\tau}})$.*

Proof. The third ratio follows from the second, since $(G_p)^{p^{\tau}}=(G^{p^{\tau}})_p$ (see also [DA]). In the sequel our arguments are based on a standard transfinite induction on τ. For this purpose, let $\tau=1$. Choose $x\in U^p(R)$. Hence $x=r^p$, where $r\in U(R)$. Since $r.r'=1$ for some fixed $r'\in R$, $r^p.r'^p=1$ and so $x\in U(R^p)$. Conversely, take $x\in U(R^p)$. Therefore $x=\alpha^p$ where $\alpha\in R$ and moreover there is an $\alpha'\in R^p$ so that $x.\alpha'=\alpha^p.\alpha'=1$. Hence

$\alpha.\alpha^{p-1}.\alpha' = 1$ and finally $\alpha \in U(R)$. Thus $x \in U^p(R)$ and the case $\tau = 1$ is complete.

Let now $\tau - 1$ exists, i.e. τ is isolated. Furthermore $U^{p^\tau}(R) = [U^{p^{\tau-1}}(R)]^p = U^p(R^{p^{\tau-1}}) = U((R^{p^{\tau-1}})^p) = U(R^{p^\tau})$.

Now assume that $\tau - 1$ does not exist, i.e. τ is a limit ordinal. It is clear that $U(L) \cap U(P) = U(L \cap P)$ where $1 \in L$, $P \leq R$, and so we derive $U^{p^\tau}(R) = \bigcap_{\sigma<\tau} U^{p^\sigma}(R) = \bigcap_{\sigma<\tau} U(R^{p^\sigma}) = U(\bigcap_{\sigma<\tau} R^{p^\sigma}) = U(R^{p^\tau})$, which gives the first equality.

Next we observe that $\mathrm{char}RG = p$ and hence by the above we have $[U(RG)]^{p^\tau} = U((RG)^{p^\tau}) = U(R^{p^\tau}G^{p^\tau})$. Clearly, $V(R^{p^\tau}G^{p^\tau}) = V(RG) \cap U(R^{p^\tau}G^{p^\tau}) = V(RG) \cap U^{p^\tau}(RG) = V^{p^\tau}(RG)$, because $V(RG)$ is an isotype subgroup in $U(RG)$ as its direct factor. $\square$

Lemma 8. ([DA], [DAN]). *$S(RG) = 1$ if and only if $N(R) = 0$ and $G_p = 1$; or $N(R) \neq 0$ and $G = 1$.*

The following lemma will be our principal tool.

Lemma 9. *For every ordinal τ and all primes $q \neq p$, the following identities are fulfilled:*

$$[GS(RG)]^{q^\tau} = G^{q^\tau}S(RG);\ [GS(RG)]^{p^\tau} = G^{p^\tau}S(R^{p^\tau}G^{p^\tau}). \tag{$\circ$}$$

$$\bigcap_{q\neq p}[G^{q^\tau}S(RG)] = (\bigcap_{q\neq p} G^{q^\tau})S(RG). \tag{$\circ\circ$}$$

Proof. ($\circ$) We will use transfinite induction on τ. First, let $\tau = 1$. Therefore $[GS(RG)]^q = G^qS^q(RG) = G^qS(RG)$ since $S(RG)$ is q-divisible, and moreover $[GS(RG)]^p = G^pS^p(RG) = G^pS(R^pG^p)$ using Lemma 7.

Now, let τ be a non-limit ordinal, i.e. $\tau - 1$ exists. Therefore

$$\begin{aligned}&[GS(RG)]^{q^\tau} = ([GS(RG)]^{q^{\tau-1}})^q = [G^{q^{\tau-1}}S^{q^{\tau-1}}(RG)]^q = [G^{q^{\tau-1}}S(RG)]^q\\ &= (G^{q^{\tau-1}})^qS^q(RG) = G^{q^\tau}S(RG),\end{aligned}$$

and besides

$$\begin{aligned}&[GS(RG)]^{p^\tau} = ([GS(RG)]^{p^{\tau-1}})^p = [G^{p^{\tau-1}}S(R^{p^{\tau-1}}G^{p^{\tau-1}})]^p\\ &= (G^{p^{\tau-1}})^pS^p(R^{p^{\tau-1}}G^{p^{\tau-1}}) = G^{p^\tau}S(R^{p^\tau}G^{p^\tau}).\end{aligned}$$

On the other hand, if τ is a limit ordinal, i.e. $\tau - 1$ does not exist, then $[GS(RG)]^{q^\tau} = \bigcap_{\sigma<\tau}[GS(RG)]^{q^\sigma} = \bigcap_{\sigma<\tau}[G^{q^\sigma}S^{q^\sigma}(RG)] = \bigcap_{\sigma<\tau}[G^{q^\sigma}S(RG)] = [\bigcap_{\sigma<\tau} G^{q^\sigma}]S(RG) = G^{q^\tau}S(RG)$.

Actually, the latter claim is valid by the following arguments: If $x \in \bigcap_{\sigma<\tau} [G^{q^\sigma} S(RG)]$, then $x = g'(r_1g_1+\cdots+r_tg_t) = g''(\alpha_1a_1+\cdots+\alpha_ta_t) = \ldots$, where $g' \in G^{q^\alpha}$, $g'' \in G^{q^\beta}$ $(\sigma\leq\alpha<\beta\leq\tau)$; $g_i, a_i \in G$; $r_i, \alpha_i \in R$ $(1\leq i\leq t)$. Since the above elements are in canonical form, then $g'g_i = g''a_i$. But $r_1g_1+\cdots+r_tg_t \in S(RG)$, hence there exists an element $g_i \in G_p$, say $g_1 \in G_p$. Besides $g'g''^{-1}$ is a p-element and obviously $a_1 \in G_p$. Finally $g'g_1 = g''a_1 \in G^{q^\beta}$ and, since we have finite support whereas the intersection is infinite because $\tau \geq \omega$, $x = g'g_1(r_1 + \cdots + r_tg_tg_1^{-1}) \in (\bigcap_{\sigma<\tau} G^{q^\sigma})S(RG)$, as claimed.
The second relation, $[GS(RG)]^{p^\tau} = \bigcap_{\sigma<\tau} [GS(RG)]^{p^\sigma} = \bigcap_{\sigma<\tau} (G^{p^\sigma} S(R^{p^\sigma} G^{p^\sigma}))$, follows from the induction hypothesis and Lemma 7.

Next we shall prove $\bigcap_{\sigma<\tau} [G^{p^\sigma} S(R^{p^\sigma} G^{p^\sigma})] = (\bigcap_{\sigma<\tau} G^{p^\sigma})(\bigcap_{\sigma<\tau} S(R^{p^\sigma} G^{p^\sigma})) = G^{p^\tau} S(R^{p^\tau} G^{p^\tau})$. Suppose $x \in \bigcap_{\sigma<\tau} [G^{p^\sigma} S(R^{p^\sigma} G^{p^\sigma})]$. Hence $x = c'(x_1c_1 + \cdots + x_tc_t) = c''(f_1b_1 + \cdots + f_tb_t) = \ldots$, where $c' \in G^{p^\alpha}$, $c'' \in G^{p^\beta}$ $(\sigma\leq\alpha<\beta\leq\tau)$; $c_i \in G^{p^\alpha}$, $b_i \in G^{p^\beta}$; $x_i \in R^{p^\alpha}$, $f_i \in R^{p^\beta}$ $(1\leq i\leq t)$. Because x is written in two canonical forms, we conclude that $x_i = f_i$ and $c'c_i = c''b_i$. Therefore we can let $c_1 \in G_p$. Finally $x = c'c_1(x_1 + \cdots + x_tc_tc_1^{-1}) \in G^{p^\beta} S(R^{p^\beta} G^{p^\beta})$ and so $x \in (\bigcap_{\sigma<\tau} G^{p^\sigma})(\bigcap_{\sigma<\tau} S(R^{p^\sigma} G^{p^\sigma}))$, because as above $c'c_1 \in \bigcap_{\sigma<\tau} G^{p^\sigma} = G^{p^\tau}$ and $x_1 + \cdots + x_tc_tc_1^{-1} \in \bigcap_{\sigma<\tau} S(R^{p^\sigma} G^{p^\sigma}) = S(R^{p^\tau} G^{p^\tau})$.

($\circ\circ$) Certainly, the left hand-side contains the right one. Conversely, suppose $x \in \bigcap_{q\neq p} [G^{q^\tau} S(RG)]$. Therefore x can be written in the form $x = \overline{g}(r_1g_1 + \cdots + r_tg_t) = \overline{\overline{g}}(\alpha_1a_1 + \cdots + \alpha_ta_t) = \ldots$, where $\overline{g} \in G^{q^\tau}$, $\overline{\overline{g}} \in G^{q_1^\tau}$; $r_i, \alpha_i \in R$; $g_i, a_i \in G$. Using the canonical form we can write $\overline{g}g_i = \overline{\overline{g}}a_i$. As above, putting $g_1 \in G_p$ we establish that $\overline{g}g_1 = \overline{\overline{g}}a_1 \in G^{q_1^\tau} \cap G^{q^\tau}$ since $a_1 \in G_p$. Finally, we obtain $x = \overline{g}g_1(r_1 + \cdots + r_tg_tg_1^{-1}) \in (\bigcap_{q\neq p} G^{q^\tau})S(RG)$, which completes the proof. □

Lemma 10. $(G^*)_p = (dG)_p = d(G_p)$.

Proof. Clearly, $G^* = G^{p^\delta}$ for some ordinal δ. Consequently $(G^*)_p = (G^{p^\delta})_p = (G_p)^{p^\delta} = d(G_p)$. On the other hand $dG \subseteq G^*$ and so $(dG)_p \subseteq (G^*)_p$. Besides $d(G_p) \subseteq dG$ and $d(G_p)$ is a p-group, i.e. $d(G_p) \subseteq G_p$. Finally $d(G_p) \subseteq (dG)_p$. □

By analogy with [F] we shall say that N is a p-nice subgroup of G if

$(G/N)^{p^\tau} = G^{p^\tau}N/N$ for each ordinal τ. It is not difficult to prove that N is p-nice in G if and only if $\bigcap_{\sigma<\tau}(G^{p^\sigma}N) = (\bigcap_{\sigma<\tau} G^{p^\sigma})N$, for every limit ordinal number τ.

Lemma 11. *The group G is p-balanced (p-nice and p-isotype) in $V(RG)$, and G_p is balanced in $S(RG)$.*

Proof. The application of Lemma 7 guarantees that $G \cap V^{p^\tau}(RG) = G \cap V(R^{p^\tau}G^{p^\tau}) = G \cap G^{p^\tau} = G^{p^\tau}$ and $G_p \cap S^{p^\tau}(RG) = G_p \cap S(R^{p^\tau}G^{p^\tau}) = G_p \cap G^{p^\tau} = {G_p}^{p^\tau}$ which insures the first assertion.

For the second assertion, as we have previously seen, it is sufficient to show only that $\bigcap_{\sigma<\tau}(G_pS^{p^\sigma}(RG)) = [\bigcap_{\sigma<\tau} S^{p^\sigma}(RG)]G_p$ and $\bigcap_{\sigma<\tau}(GV^{p^\sigma}(RG)) = [\bigcap_{\sigma<\tau} V^{p^\sigma}(RG)]G$, for any limit ordinal τ. But from Lemma 7 it follows at once that $S^{p^\sigma}(RG) = S(R^{p^\sigma}G^{p^\sigma})$ and $V^{p^\sigma}(RG) = V(R^{p^\sigma}G^{p^\sigma})$, hence our conditions can be equivalently modified to $\bigcap_{\sigma<\tau}[G_pS(R^{p^\sigma}G^{p^\sigma})] = [\bigcap_{\sigma<\tau} S(R^{p^\sigma}G^{p^\sigma})]G_p = S(R^{p^\tau}G^{p^\tau})G_p$ and $\bigcap_{\sigma<\tau}[GV(R^{p^\sigma}G^{p^\sigma})] = [\bigcap_{\sigma<\tau} V(R^{p^\sigma}G^{p^\sigma})]G = V(R^{p^\tau}G^{p^\tau})G$. We will argue only the first equality. The proof of the second equality is similar.

Suppose that $x \in \bigcap_{\sigma<\tau}[G_pS(R^{p^\sigma}G^{p^\sigma})]$, hence $x = g_p(x_1c_1 + \cdots + x_tc_t) = g'_p(f_1b_1 + \cdots + f_tb_t) = \ldots$, where $g_p, g'_p \in G_p$; $x_i \in R^{p^\alpha}$, $c_i \in G^{p^\alpha}$; $f_i \in R^{p^\beta}$, $b_i \in G^{p^\beta}$ $(\sigma \le \alpha < \beta \le \tau;\ 1 \le i \le t)$. The canonical form of x yields $g_pc_i = g'_pb_i$ and $x_i = f_i$, where we may presume that $c_1 \in G_p$. Therefore $x = g_pc_1(x_1 + \cdots + x_tc_tc_1^{-1}) \in G_pS(R^{p^\beta}G^{p^\beta})$. Finally, since the support is finite while the relationships are not, we have $x \in G_pS(R^{p^\tau}G^{p^\tau})$. □

Remark. The *"nice"* property of certain subgroups of $S(RG)$ and $V(RG)$ was also considered by May [M], but when R is a field. The preceding lemma generalizes the May's attainment for niceness in [M] over an arbitrary ring.

Lemma 12. *If N is a p-balanced subgroup of G, then*

$$(G/N)^{p^\tau} \cong G^{p^\tau}/N^{p^\tau},\ \textit{for each ordinal}\ \tau;$$

$$(G/N)^* = (G^*N)/N \cong G^*/N^*.$$

Proof. We shall verify only the first isomorphism, since the second is its immediate consequence. By the supposition and using the well-known first theorem of Noether for the isomorphism we have $(G/N)^{p^\tau} = G^{p^\tau}N/N \cong G^{p^\tau}/(N \cap G^{p^\tau}) = G^{p^\tau}/N^{p^\tau}$. □

Proof of Proposition 1. The proof follows directly from Lemma 11 and Lemma 12 by invoking to the above listed May's proposition which assures that $V(FG)/G \cong S(FG)/G_p$. □

Proof of Corollary 2. Evidently $S(RG)$ is reduced if and only if, by Lemma 7, $dS(RG) = S(R_dG^*) = 1$, i.e. we can employ Lemma 8 to infer that the first part holds. For the second assertion, it is clear that $S(RG)/G_p$ is reduced if and only if $d[S(RG)/G_p] = 1$, i.e. applying Proposition 1, $S(R_dG^*) = (G^*)_p$. This is equivalent to $G^* \neq (G^*)_p = 1$ and $N(R_d) = 0$, or $G^* = (G^*)_p \neq 1$, $|G^*| = 2$ and $|R_d| = 2$, or $G^* = 1$. □

Proof of Theorem 3. Following [F], $V^\tau(FG) = \bigcap\limits_q V^{q^{\omega\tau}}(FG) = \bigcap\limits_{q\neq p} V^{q^{\omega\tau}}(FG) \cap V^{p^{\omega\tau}}(FG)$, then using the May's proposition cited above, we have $V^\tau(FG) = \bigcap\limits_{q\neq p} [GS(FG)]^{q^{\omega\tau}} \cap V^{p^{\omega\tau}}(FG)$. On the other hand Lemma 9 yields that $V^\tau(FG) = \bigcap\limits_{q\neq p} [G^{q^{\omega\tau}} S(FG)] \cap V^{p^{\omega\tau}}(FG) = ([\bigcap\limits_{q\neq p} G^{q^{\omega\tau}}]S(FG)) \cap V^{p^{\omega\tau}}(FG)$. Now, exploiting Lemma 7, we get $V^\tau(FG) = ([\bigcap\limits_{q\neq p} G^{q^{\omega\tau}}]S(FG)) \cap V(F^{p^{\omega\tau}}G^{p^{\omega\tau}})$. If we show that the last intersection is equal to $(\bigcap\limits_q G^{q^{\omega\tau}})$ $S(F^{p^\omega}G^{p^{\omega\tau}})$ then the proof will be complete.

In fact, first of all, $F^{p^{\omega\tau}} = F^{p^\omega}$. Secondly, take x to lie in the left hand-side of the wanted equality $F^{p^{\omega\tau}}$. Hence $x = a(f_1g_1 + \cdots + f_tg_t) = \alpha_1b_1 + \cdots + \alpha_tb_t$, where $a \in \bigcap\limits_{q\neq p} G^{q^{\omega\tau}}$, $g_i \in G$, $b_i \in G^{p^{\omega\tau}}$; $f_i \in F$, $\alpha_i \in F^{p^\omega}$ $(1\leq i\leq t)$. Therefore $f_i = \alpha_i$ and $ag_i = b_i$ for each i. Set $x = ag_1(f_1 + \cdots + f_tg_tg_1^{-1})$. We may assume that $g_1 \in G_p$. So $ag_1 \in \bigcap\limits_{q\neq p} G^{q^{\omega\tau}}$ and obviously $ag_1 \in \bigcap\limits_q G^{q^{\omega\tau}}$. Therefore, $f_1 + \cdots + f_tg_tg_1^{-1} = g_1^{-1}(f_1g_1 + \cdots + f_tg_t) \in S(FG) \cap F^{p^\omega}G^{p^{\omega\tau}} = S(F^{p^\omega}G^{p^{\omega\tau}})$. Finally, $x \in (\bigcap\limits_q G^{q^{\omega\tau}})S(F^{p^\omega}G^{p^{\omega\tau}})$. □

Proof of Corollary 4. Using Lemma 7, it is a routine matter to check that $dS(FG) = S(F_dG^*)$, where $F_d = F^{p^\omega}$. Let τ be the smallest i.e. the first ordinal with the property that $V^\tau(FG) = V^{\tau+1}(FG)$, $G^\tau = G^{\tau+1}$ and $G^{p^\tau} = G^{p^{\tau+1}}$. Hence $dV(FG) = V^\tau(FG), dG = G^\tau$ and $G^* = G^{p^\tau} = G^{p^{\omega\tau}}$. Thus Theorem 3 is applicable and the proof is complete. □

Proof of Corollary 5. Since dG is divisible, then it is a direct factor of $dV(FG)$ utilizing [F], i.e. $dV(FG) \cong dG \times dV(FG)/dG$. But Corollary 4 implies that $dV(FG)/dG = dGS(F_dG^*)/dG \cong S(F_dG^*)/(dG)_p$. Furthermore we can apply Lemma 10 and Lemma 12 to obtain the result. $\square$

Proof of Corollary 6. If $V(FG)$ is reduced, then clearly G, being its subgroup, is also reduced. Conversely, let us assume that G is reduced. Hence $dG = 1$ and so $(dG)_p = d(G_p) = (G^*)_p = 1$ by making use of Lemma 10. Therefore, by a direct application of Corollary 5, or by virtue of Corollaries 2 and 4, we deduce that $dV(FG) = 1$, i.e. $V(FG)$ is reduced. $\square$

Remark. Using the preceding theorem of Nachev for the description of $dS(RG)$ and going from Theorem 3 to Corollary 5 along with some standard group-theoretic facts, given in [F], we can verify that $dV(FG)$ is completely characterized, provided $tG = G_p$.

In order to extract the explicit isomorphism relationship, we observe that since $dS(FG)$ is divisible,

$$dV(FG) \cong dS(FG) \times dV(FG)/dS(FG) \cong dS(FG) \times dG/d(G_p),$$

where the second isomorphism holds by Corollary 4.

Thus, for a p-mixed group G,

$$dV(FG) \cong \coprod_{\lambda} \mathbf{Z}\,(p^\infty) \times \coprod_{r_0 d(G)} \mathbb{Q},$$

where $\lambda = max(|F^{p^\omega}|, |G^*|)$ if $(G^*)_p \neq 1$ or $\lambda = 0$ otherwise, and $\mathbb{Q}$ is regarded as an additive group.

III. *The description of $dV(FG)$ for $G = G_p \times G/G_p$*

Let G be p-splitting and F an algebraically closed field. Then, from the results in [D] and [DANC], it follows that

$$dV(FG) \cong \Big(\coprod_{\mu} d(G/G_t)\Big) \times \Big(\coprod_{\mu} F^*\Big) \times S(FG^*)\,,$$

where $\mu = |tG/G_p| \geq \aleph_0$ or $\mu = |tG/G_p| - 1$ otherwise. Thus this group is completely characterized up to an isomorphism.

Now suppose G is p-splitting (i.e. $G = G_p \times G/G_p$) and F is arbitrary. Then, owing to ([F], p.124, Theorem 23.1) and to our results in [D] and [DANC], we establish that

$$dV(FG) \cong \coprod_{r_0 dV(F(G/G_p))} \mathbb{Q} \times \coprod_{q \neq p} [\mathop{\times}_{r_q dV(F(G/G_p))} \mathbf{Z}\,(q^\infty)] \times \coprod_{\lambda} \mathbf{Z}\,(p^\infty)\,,$$

where λ is calculated as in the Nachev's theorem stated above in paragraph 1. Thus, if we compute the cardinal numbers $r_0 dV(F(G/G_p))$ (when G splits $r_0 dV(F(G/G_p))$ and even $r_0 V(F(G/G_p))$ were calculated in [D] and [DANCH]) and $r_q dV(F(G/G_p))$, then the structure of $dV(FG)$, under these restrictions, will be completely determined. Thus, in accordance with the latter formula, in the case when G is a splitting group, the cardinals $r_q dV(F(G/G_p))$ are only needed for the isomorphic classification of $dV(FG)$. However, their computation is a problem of some other investigation where a new approach might work.

We close this paper with some open problems and questions.

4. Concluding discussion

Here is a question which immediately arises. What is the structure of $dV(FG)$ in the general case (in particular when $tG \neq G_p$ and tG/G_p is finite; or $G = tG$)? When G is p-splitting (in particular torsion), it follows from [D] and [DANC], that $dV(FG) = dV(F(G/G_p)) \times S(F_d G^*)$ and thus it remains only to describe $dV(F(G/G_p))$ in the terms of F and G. But when G is torsion and F^* is torsion (i.e. F is an algebraic extension of a finite field), employing [DANCH], $dV(F(G/G_p))$ is torsion whence $dV(FG) = \coprod_{q \neq p} dV_q(F(G/G_p)) \times S(F_d G^*)$. Thus $dV_q(F(G/G_p))$ need be classified.

Acknowledgments. The author would like to thank the referee for the careful reading of the paper and the helpful expert comments.

REFERENCES

[D] **P.V. Danchev,** *Units in Abelian group rings of prime characteristic,* C. R. Acad. Bulg. Sci., 48 (8) (1995), 5–8.

[DA] ———— **,** *Topologically pure and basis subgroups in commutative group rings,* C. R. Acad. Bulg. Sci., 48 (9–10) (1995), 7–10.

[DAN] ———— **,** *Commutative group algebras of σ -summable Abelian groups,* Proc. Amer. Math. Soc., 125 (9) (1997), 2559–2564.

[DANC] ———— **,** *Normed units in Abelian group rings,* Glasgow Math. J., 43 (3) (2001), 365–373.

[DANCH] ———— **,** *Criteria for unit groups in commutative groups rings,* to appear.

[DANCHEV] ———— **,** *Sylow p-subgroups of commutative modular and semisimple group rings,* C. R. Acad. Bulg. Sci., 54 (6) (2001), 5–6.

[F] L. Fuchs, *Infinite Abelian Groups,* I–II, Moscow, Mir, 1974–1977. (In Russian.)

[M] W. May, *The direct factor problem for modular Abelian group algebras,* Contemp. Math., 93 (1989), 303–308.

[N] N.A. Nachev, *Invariants of the Sylow p-subgroup of the unit group of commutative group ring of characteristic p,* C. R. Acad. Bulg. Sci., 47 (1) (1994), 9–12.

[NA] **N.A. Nachev,** *Invariants of the Sylow p-subgroup of the unit group of a commutative group ring of characteristic p,* Commun. Algebra, 23 (7) (1995), 2469–2489.

(Received: July 6, 2004)
(Revised: August 31, 2004)

Department of Algebra
Plovdiv University
4000 Plovdiv
Bulgaria BGR
E-mail: pvdanchev@yahoo.com

Maksimalne dijeljive podgrupe u modularnim grupnim algebrama p–miješanih i p–podijeljenih Abelovih grupa

Peter V. Danchev

Sadržaj

Neka je FG grupna algebra Abelove grupe G nad poljem F sa $\mathrm{char} F = p \neq 0$. Glavni cilj i rezultat ovog rada je izračunavanje maksimalne dijeljive podgrupe normirane jedinične grupe $V(FG)$ u FG u slučaju kada je torzioni dio tG od G p-primarni. Kao korolar se pokazuje da je $V(FG)$ reducirana, ako je i samo ako G reducirana, pod uslovom da je tG p-torzija. Takodjer se razmatra struktura maksimalne dijeljive podgrupe od $V(FG)$ kada je G p-podijeljena. Ovo proširuje rezultate N. Nacheva [Na].