New lower bounds for two multicolor classical Ramsey numbers

H. Luo, W. Su (China) and Y.-Q. Shen (USA)

Abstract. We present an algorithm to find lower bounds for multicolor classical Ramsey numbers by using 2-normalized cyclic graphs of prime order, and use it to obtain new lower bounds for two multicolor classical Ramsey numbers: \( R(3, 3, 12) \geq 182 \), \( R(3, 3, 13) \geq 212 \).

1. Introduction

The multicolor classical Ramsey number \( R(q_1, q_2, \ldots, q_n) \) is the smallest integer \( r \) such that if the edges of \( K_r \), the complete graph of order \( r \), are colored with \( n \) colors, there exists a monochromatic \( K_{q_i} \) for some \( i \). The definition is well defined and the existence of these numbers has been proved (See the book by Graham, Rothschild and Spencer [1]). However for concrete \( q_i \)'s, only very few actual Ramsey numbers are known. Radziszowski [4], in his dynamic survey, lists the known Ramsey numbers and the updated upper and lower bounds.

In [3], we presented an algorithm based on the properties of normalized cyclic graphs with prime order and use it to obtain several new lower bounds on two-color Ramsey numbers. The method reduces to a certain amount of computation depending on the sizes of the parameter sets.

In this paper, we will present an algorithm for lower bounds of three-color Ramsey numbers which improves the effectiveness of the previous algorithm. Our new method is based on some properties of 2-normalized cyclic graphs of prime order. The method reduces to a certain amount of computation for parameter sets of any size. Using our new algorithm we obtain:

\[
R(3, 3, 12) \geq 182, \quad R(3, 3, 13) \geq 212.
\] (1)

2000 Mathematics Subject Classification: 05C55.

Keywords and phrases: Multicolor Ramsey numbers, normalized cyclic graphs of prime order, 2-normalized cyclic graphs of prime order, parameter sets of cyclic graphs.
Our results improve the known bounds $R(3,3,12) \geq 181, R(3,3,13) \geq 205$ listed in [4]. The algorithm also reduces the amount of computation necessary in finding new lower bounds for classical Ramsey numbers.

2. 2–normalized cyclic graphs of prime order

Given a prime number $p = 2m + 1$, let $Z_p = \{−m, \ldots, −1, 0, 1, \ldots m\} = [−m, m]$. We write $s \equiv t$ in $Z_p$ if and only if $s \equiv t \pmod{p}$. For any parameter set $S \subset Z_p^+ = [1, m]$, the cyclic graph $G_p(S)$ of order $p$ with the parameter set $S$ is $G_p(S) = (V, E) = (Z_p, E)$, where $E = \{x, y : |x − y| \in S\}$. Using cyclic graphs of prime order to obtain lower bounds for classical Ramsey numbers has been successful in the past [4]. For special constructions of parameter sets to reduce the computation, see [3], [5], [6], [7]. In [3], we use normalized cyclic graphs of prime order to reduce the amount of computation.

Let $g$ be a primitive root of $p$ and $k = |S| \geq 2$. A cyclic graph of order $p$ with $S$ is called normalized if the following three conditions are satisfied:

\begin{align}
S &= \{|g^{a_0}|, |g^{a_1}|, \ldots, |g^{a_{k-1}}| \in Z_p^+ : a_j \in [0, m - 1]\}, \\
0 &= a_0 < a_1 < \ldots < a_{k-1} \leq m - a_1, \\
a_1 &= \text{min}\{a_j - a_{j-1} : j \in [1, k - 1]\} \leq m/k.
\end{align}

We call the set $B(S) \equiv \{a_0, a_1, \ldots, a_{k-1}\} \subset [0, m - 1]$ the corresponding subset associated with $S$. In [3] we have proved the following theorem:

**Theorem 1.** Any cyclic graph $G_p(S)$ with $|S| \geq 2$ is isomorphic to a normalized $G_p(S^*)$ for some parameter set $S^* \subset Z_p^+$ with $|S^*| = |S|$.

For a given prime number $p$, and a fixed integer $k$, the naive approach require computation over all parameter sets $S$ with $|S| = k$ to search for an effective parameter set to produce a lower bound. Theorem 1 tells us that we only need to restrict our search to the parameter sets of normalized cyclic graphs. The total number of possible parameter sets of normalized cyclic graphs compared to all the possible parameter sets of cyclic graphs when the size $|S| = k$ is fixed depends on the size $k$. Therefore a large portion of the naive method of searching over all of the subsets size $k$ can be saved only when $k$ is small compared to $m$ in [3]. However, we have found that if 2 is a primitive root of the prime number $p$, and the cyclic graphs do not contain the complete subgraph $K_3$, then the total number of possible parameter sets of normalized cyclic graphs is less than 25% of the parameter sets of cyclic graphs in the worst cases no matter what size $k$ is.
A normalized cyclic graph of prime order \( p \) is said to be \( 2\text{-normalized} \) if it satisfies
\[
a_1 \geq 2. \tag{5}
\]

Now we present a theorem which gives a sufficient condition for a cyclic graph to be isomorphic to a \( 2\text{-normalized} \) cyclic graph:

**Theorem 2.** If \( g = 2 \) is a primitive root of the prime number \( p \), and \( G_p(S) \) contains no complete subgraph \( K_3 \), then this cyclic graph of prime order \( p \) is isomorphic to a \( 2\text{-normalized} \) cyclic graph of order \( p \).

**Proof.** Using Theorem 1, \( G_p(S) \) is isomorphic to some normalized cyclic graph \( G_p(S^*) \) with \( |S| = |S^*| \). Now we only need to show that if \( g = 2 \) is a primitive root of \( p \), then \( a_1 \geq 2 \) for \( S^* \). If not, \( a_1 = 1 \), then \( S^* = \{2^0, 2^1, \ldots\} = \{1, 2, \ldots\} \) which implies that the cyclic graph with parameter \( S^* \) contains a complete \( K_3 \) with vertices \( \{0, 1, 2\} \), contradicting the isomorphism of the two graphs since the original cyclic graph does not contain \( K_3 \). \( \square \)

### 3. Counting \( 2\text{-normalized} \) cyclic graphs

In the last section, we proved that if a normalized cyclic graph does not contain the complete subgraph \( K_3 \) and \( g = 2 \) is a primitive root of the prime number \( p \), then the normalized cyclic graph is \( 2\text{-normalized} \). We now give an estimate for the percentage of \( 2\text{-normalized} \) graphs among all cyclic graphs when \( m, k \) are fixed. Denote the set of all parameter sets \( S \) with \( |S| = k \) as \( W(m, k) \) when \( p \) and \( k \) are fixed by counting their parameter sets. We compare \( |W(m, k)| \) with \( \binom{m}{k} \), the number of all parameter sets with size \( k \) in naive computation. We have the following theorem, which compares to a similar estimate in Theorem 4.2 [3], but it is better since our last estimate in (b) is independent of the size \( k \).

**Theorem 3.** The number of parameter sets with \( |S| = k \) satisfies the following estimates:
(a) \( |W(m, k)| = \sum_{a_1=2}^{m} \binom{m+k-2-a_1k}{k-2} \).
(b) \( |W(m, k)| / \binom{m}{k} \leq \frac{m-1}{4m} \prod_{s=2}^{k-1} \left(1 - \frac{k}{m+s}\right) < \frac{1}{4} \).

**Proof.** (a) We count the parameter sets with fixed size \( k \) and \( m \) by using their corresponding subsets \( B(S) \) in \([0, m-1]\). From (3), (4): \( 2 \leq a_1 \leq \lfloor \frac{m}{k} \rfloor \). Let \( a_j = a_{j-1} = a_1 + c_j, j = 2, \ldots, k \), where \( c_j \geq 0 \). Then \( \sum_{j=2}^{k} c_j = m - a_1 - a_1(k-1) = m - a_1k \). When \( k \) is fixed, \(|W(m, k)|\) is equal to the
number of all possibilities of distributing \( m - a_k \) ones into \( k - 1 \) places on a line. That can be calculated by

\[
|W(m,k)| = \sum_{a_1=2}^{\frac{m}{k}} \binom{(m-a_1)+(k-1)-1}{m-a_1} = \sum_{a_1=2}^{\frac{m}{k}} \binom{m+k-2-a_1}{k-2}.
\]

(b) Using (a), we have

\[
|W(m,k)|/\binom{m}{k} \leq \sum_{a_1=2}^{\frac{m}{k}} \binom{m-k-2}{k-2} / \binom{m}{k} \leq \frac{m-k}{k} \binom{m-k-2}{k-2} / \binom{m(m-1)}{k(k-1)} \binom{m-2}{k-2}
\]

\[
= \frac{(m-k)(k-1)(m-k-2)(m-k-3)...(m-2k+1)}{m(m-1)(m-2)(m-3)...(m-k+1)} \leq \frac{(1/4)(m-1)^2}{m(m-1)} \prod_{s=2}^{k-1} \left( \frac{m-s-k}{m-s} \right)
\]

\[
= \frac{m-1}{4m} \prod_{s=2}^{k-1} \left( 1 - \frac{k}{m-s} \right)
\]

using that the maximum of \((m-k)(k-1)\) is \((1/4)(m-1)^2\) when \(k = (m+1)/2\). So the first part of (b) is proved. The second part of (b) is obvious. The proof is completed. □

From the theorem, in the worst cases, at least 75% of the parameter sets can be omitted from checking. The example in the next section shows that in the actual computation, there are more parameter sets which can be omitted.

4. An algorithm to finding lower bounds for three–color Ramsey numbers

Based on Theorem 2, we present an algorithm for a lower bound for \(R(3,3,q_3)\). The algorithm can be extended to find lower bounds for more general multicolor classical Ramsey numbers with some modifications.

**Algorithm 1.** For given \(q_1 = q_2 = 3, q_3 \geq 3\), perform the following steps:

(a) Choose a prime number \(p = 2m + 1\) with primitive root \(g = 2\), and positive integers \(k_1, k_2, k_3 = m - k_1 - k_2\).

(b) For \(i = 1, 2\), do Steps (c) - (g):
(c) Let \( M_1 = [0, m - 2] \) and \( M_2 = [1, m - 1] - B_1 \).
(d) List all possible subsets \( B_{i,j_i} \) in \( M_i \) with \( k_i \) elements according to the lexicographic order of their elements. If \( i = 1 \), list all subsets \( B_{1,j_1} \) with \( k_1 \) elements using the restrictions (2) – (5) of the 2-normalized cyclic graph. For each \( j_i \), do Steps (e) and (f):
(e) Use \( B_{i,j_i} \) to form \( S_{i,j_i} \).
(f) Compute the clique number \( [S_{i,j_i}] \) of \( G_p(S_{i,j_i}) \). If \( [S_{i,j_i}] \leq q_i \), then let \( S_i = S_{i,j_i} \) and \( B_i = B_{i,j_i} \). Go to (b) if \( i = 1 \), and go to (h) if \( i = 2 \), then let \( i = i + 1 \).
(g) Conclude that for the chosen \( p, k_1, k_2 \), the method cannot produce a lower bound. Stop.
(h) Let \( S_3 = \mathbb{Z}_p^+ - \cup_{i=1}^2 S_i \), compute the clique number \( [S_3] \). If \( [S_3] \leq q_3 \), go to (i).
(i) Conclude that \( R(3, 3, q_3) \geq p + 1 \). Stop.

Note that in practice, for the chosen \( p, k_1, k_2 \) the algorithm stops at Step (g) frequently, therefore we need to choose another set of \( p, k_1, k_2 \) and do the same steps in the algorithm again. The correct terms \( p, k_1, k_2 \) for effective parameter sets are usually obtained after many experiments. In general, we need to select a prime number \( p \), not too small to produce a new lower bound, and not too large so the amount of calculations can be performed in a computer.

We present an example to illustrate our algorithm. The lower bound was initially obtained in [2].

**Example 1.** \( R(3, 3, 4) \geq 30 \).

In this case \( q_3 = 4 \). After experimenting with different choices, we pick \( p = 29 \) in Step (a) that gives \( g = 2 \) and \( m = 14 \), and choose \( k_1 = k_2 = 4 \), so \( k_3 = 6 \). There are 34 subsets \( B_{1,j_1} \) in \( [0, 14 - a_1] \) fitting \( 2 \leq a_1 \leq 14/4 \) with 4 elements:

\[
\begin{align*}
{0, 2, 4, 6}, & \{0, 2, 4, 7\}, \{0, 2, 4, 8\}, \{0, 2, 4, 9\}, \{0, 2, 4, 10\}, \{0, 2, 4, 11\}, \\
& \{0, 2, 4, 12\}, \{0, 2, 5, 7\}, \{0, 2, 5, 8\}, \{0, 2, 5, 9\}, \{0, 2, 5, 10\}, \{0, 2, 5, 11\}, \\
& \{0, 2, 6, 12\}, \{0, 2, 6, 8\}, \{0, 2, 6, 9\}, \{0, 2, 6, 10\}, \{0, 2, 6, 11\}, \{0, 2, 6, 12\}, \\
& \{0, 2, 7, 9\}, \{0, 2, 7, 10\}, \{0, 2, 7, 11\}, \{0, 2, 7, 12\}, \{0, 2, 8, 10\}, \{0, 2, 8, 11\}, \\
& \{0, 2, 8, 12\}, \{0, 2, 9, 11\}, \{0, 2, 9, 12\}, \{0, 2, 10, 12\}, \{0, 3, 6, 9\}, \{0, 3, 6, 10\}, \\
& \{0, 3, 6, 11\}, \{0, 3, 7, 10\}, \{0, 3, 7, 11\}, \{0, 3, 8, 11\}.
\end{align*}
\]

After Steps (c) and (d), we obtain the subset \( S_1 = \{4, 10, 12\} \) which fits the requirement, i.e. the clique number is \( [S_1] = 2 < 3 = q_1 \). Using naive computation to find a suitable subset with 4 elements from 14 positions in \( \{1, 2, \ldots , 14\} \) would require checking \( \binom{14}{4} = 1001 \) subsets. Using our algorithm, we find three parameter sets:
$S_1 = \{1, 4, 10, 12\}, \ S_2 = \{2, 5, 8, 9\}, \ S_3 = \{3, 6, 7, 11, 13, 14\}$

with three clique numbers respectively $[S_1] = 2, \ [S_2] = 2, \ [S_3] = 3$, therefore $R(3,3,4) \geq 30$.

The example shows, in actual computation the real percentage of the parameter sets needing to be checked to obtain $S_1$ is much less than 25%, which is the conservative estimate from Theorem 3. The percentage in the example is $\frac{34}{1001} \approx 3.4\%$.

5. Two new lower bounds for Ramsey numbers

Using our algorithm, we have found two sets of effective parameter sets, and these sets yield the new lower bounds in (1) that yield the following theorem:

**Theorem 4.** $R(3,3,12) \geq 182, \ R(3,3,13) \geq 212$.

**Proof.** For each $p$, with primitive root $g = 2$, we use Algorithm 1 to obtain the parameter sets $S_1, S_2, S_3$ and their related clique numbers. We list them as follows:

(a) Let $p = 181$ which gives $g = 2$. When $S_1 = \{1, 4, 7, 19, 22, 32, 35, 45, 50, 53, 56, 65, 76, 79, 89\}$, $S_2 = \{2, 3, 9, 10, 17, 24, 25, 31, 38, 39, 46, 57, 61, 68, 72, 73, 80, 87\}$, we obtain clique numbers $[S_1] = [S_2] = 2, \ [S_3] = 11$ which implies that $R(3,3,12) \geq 182$.

(b) Let $p = 211$ which gives $g = 2$. When $S_1 = \{1, 4, 6, 9, 23, 25, 40, 55, 58, 60, 68, 71, 73, 76, 87, 90, 92, 97, 102, 104\}$, $S_2 = \{8, 11, 12, 13, 18, 27, 28, 32, 42, 48, 49, 63, 65, 72, 79, 82, 86, 88, 89, 103\}$, we obtain clique numbers $[S_1] = [S_2] = 2, \ [S_3] = 12$ which implies that $R(3,3,13) \geq 212$. \hfill \Box

**Acknowledgements.** The first two authors were partially supported by the National Natural Science Foundation of China (10161003) and Guangxi Natural Science Foundation of China (GKZ0447010).

**REFERENCES**


New lower bounds for two multicolor classical Ramsey numbers


(Received: April 11, 2004) Haipeng Luo
(Revised: October 8, 2004) Guangxi Academy of Science
Wenlong Su
Nanning, Guangxi 530031, China
Yun-Qiu Shen
Guangxi Computer Center
Nanning, Guangxi 530022, China
Department of Mathematics
Western Washington University
Bellingham, WA 98225, USA
E-mail: yqshen@cc.wwu.edu

Nove donje granice dvaju multikolornih klasičnih Ramsey brojeva

H. Luo, W. Su i Y.-Q. Shen

Sadržaj

U radu se prezentira algoritam za iznalaženje donjih granica multikolornih klasičnih Ramsey brojeva uz korištenje 2-normaliziranih cikličkih grafova prvog reda. Dobivaju se nove donje granice dvaju multikolornih klasičnih Ramsey brojeva: $R(3,3,12) \geq 182$, $R(3,3,13) \geq 212$. 