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New lower bounds for two multicolor classical Ramsey numbers

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Abstract. We present an algorithm to find lower bounds for multicolor classical Ramsey numbers by using 2-normalized cyclic graphs of prime order, and use it to obtain new lower bounds for two multicolor classical Ramsey numbers: $R(3,3,12) \geq 182$, $R(3,3,13) \geq 212$.

1. Introduction

The multicolor classical Ramsey number $R(q_1, q_2, ...q_n)$ is the smallest integer r such that if the edges of K_r, the complete graph of order r, are colored with n colors, there exists a monochromatic K_{q_i} for some i. The definition is well defined and the existence of these numbers has been proved (See the book by Graham, Rothschild and Spencer [1]). However for concrete $q_i's$, only very few actual Ramsey numbers are known. Radziszowski [4], in his dynamic survey, lists the known Ramsey numbers and the updated upper and lower bounds.

In [3], we presented an algorithm based on the properties of normalized cyclic graphs with prime order and use it to obtain several new lower bounds on two-color Ramsey numbers. The method reduces to a certain amount of computation depending on the sizes of the parameter sets.

In this paper, we will present an algorithm for lower bounds of three-color Ramsey numbers which improves the effectiveness of the previous algorithm. Our new method is based on some properties of 2-normalized cyclic graphs of prime order. The method reduces to a certain amount of computation for parameter sets of any size. Using our new algorithm we obtain:

$$R(3,3,12) \geq 182, \quad R(3,3,13) \geq 212. \tag{1}$$

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Our results improve the known bounds $R(3,3,12) \geq 181, R(3,3,13) \geq 205$ listed in [4]. The algorithm also reduces the amount of computation necessary in finding new lower bounds for classical Ramsey numbers.

2. 2–normalized cyclic graphs of prime order

Given a prime number $p = 2m+1$, let $Z_p = \{-m, ..., -1, 0, 1, ...m\} = [-m, m]$. We write $s = t$ in Z_p if and only if $s \equiv t (\mathrm{mod}\ p)$. For any parameter set $S \subset Z_p^+ = [1, m]$, the cyclic graph $G_p(S)$ of order p with the parameter set S is $G_p(S) = (V, E) = (Z_p, E)$, where $E = \{\{x, y\} : |x - y| \in S\}$. Using cyclic graphs of prime order to obtain lower bounds for classical Ramsey numbers has been successful in the past [4]. For special constructions of parameter sets to reduce the computation, see [3], [5], [6], [7]. In [3], we use normalized cyclic graphs of prime order to reduce the amount of computation.

Let g be a primitive root of p and $k = |S| \geq 2$. A cyclic graph of order p with S is called *normalized* if the following three conditions are satisfied:

$$S = \{|g^{a_0}|, |g^{a_1}|, ..., |g^{a_{k-1}}| \in Z_p^+ : a_j \in [0, m-1]\}, \tag{2}$$

$$0 = a_0 < a_1 < ... < a_{k-1} \leq m - a_1, \tag{3}$$

$$a_1 = min\{a_j - a_{j-1} : j \in [1, k-1]\} \leq m/k. \tag{4}$$

We call the set $B(S) \equiv \{a_0, a_1, ..., a_{k-1}\} \subset [0, m-1]$ the corresponding subset associated with S. In [3] we have proved the following theorem:

Theorem 1. *Any cyclic graph $G_p(S)$ with $|S| \geq 2$ is isomorphic to a normalized $G_p(S^*)$ for some parameter set $S^* \subset Z_p^+$ with $|S^*| = |S|$.*

For a given prime number p, and a fixed integer k, the naive approach require computation over all parameter sets S with $|S| = k$ to search for an effective parameter set to produce a lower bound. Theorem 1 tells us that we only need to restrict our search to the parameter sets of normalized cyclic graphs. The total number of possible parameter sets of normalized cyclic graphs compared to all the possible parameter sets of cyclic graphs when the size $|S| = k$ is fixed depends on the size k. Therefore a large portion of the naive method of searching over all of the subsets size k can be saved only when k is small compared to m in [3]. However, we have found that if 2 is a primitive root of the prime number p, and the cyclic graphs do not contain the complete subgraph K_3, then the total number of possible parameter sets of normalized cyclic graphs is less than 25% of the parameter sets of cyclic graphs in the worst cases no matter what size k is.

A normalized cyclic graph of prime order p is said to be *2-normalized* if it satisfies

$$a_1 \geq 2. \tag{5}$$

Now we present a theorem which gives a sufficient condition for a cyclic graph to be isomorphic to a 2-normalized cyclic graph:

Theorem 2. *If $g = 2$ is a primitive root of the prime number p, and $G_p(S)$ contains no complete subgraph K_3, then this cyclic graph of prime order p is isomorphic to a 2-normalized cyclic graph of order p.*

Proof. Using Theorem 1, $G_p(S)$ is isomorphic to some normalized cyclic graph $G_p(S^*)$ with $|S| = |S^*|$. Now we only need to show that if $g = 2$ is a primitive root of p, then $a_1 \geq 2$ for S^*. If not, $a_1 = 1$, then $S^* = \{2^0, 2^1, ...\} = \{1, 2, ...\}$ which implies that the cyclic graph with parameter S^* contains a complete K_3 with vertices $\{0, 1, 2\}$, contradicting the isomorphism of the two graphs since the original cyclic graph does not contain K_3. $\square$

3. Counting 2–normalized cyclic graphs

In the last section, we proved that if a normalized cyclic graph does not contain the complete subgraph K_3 and 2 is a primitive root of the prime number p, then the normalized cyclic graph is 2-normalized. We now give an estimate for the percentage of 2-normalized graphs among all cyclic graphs when m, k are fixed. Denote the set of all parameter sets S with $|S| = k$ as $W(m, k)$ when p and k are fixed by counting their parameter sets. We compare $|W(m, k)|$ with $\binom{m}{k}$, the number of all parameter sets with size k in naive computation. We have the following theorem, which compares to a similar estimate in Theorem 4.2 [3], but it is better since our last estimate in (b) is independent of the size k.

Theorem 3. *The number of parameter sets with $|S| = k$ satisfies the following estimates:*

(a) $|W(m,k)| = \sum_{a_1=2}^{\lfloor \frac{m}{k} \rfloor} \binom{m+k-2-a_1k}{k-2}$.

(b) $|W(m,k)| \Big/ \binom{m}{k} \leq \frac{m-1}{4m} \prod_{s=2}^{k-1} \left(1 - \frac{k}{m-s}\right) < \frac{1}{4}$.

Proof. (a) We count the parameter sets with fixed size k and m by using their corresponding subsets $B(S)$ in $[0, m-1]$. From (3), (4): $2 \leq a_1 \leq \lfloor \frac{m}{k} \rfloor$. Let $a_j - a_{j-1} = a_1 + c_j, j = 2, ...k$, where $c_j \geq 0$. Then $\sum_{j=2}^{k} c_j = m - a_1 - a_1(k-1) = m - a_1k$. When k is fixed, $|W(m,k)|$ is equal to the number of all possibilities of distributing $m - a_1 k$ ones into $k-1$ places on a line. That can be calculated by

$$|W(m,k)| = \sum_{a_1=2}^{\lfloor \frac{m}{k} \rfloor} \binom{(m-a_1k)+(k-1)-1}{m-a_1k} = \sum_{a_1=2}^{\lfloor \frac{m}{k} \rfloor} \binom{m+k-2-a_1k}{k-2}.$$

(b) Using (a), we have

$$\begin{aligned}
|W(m,k)| \Big/ \binom{m}{k} &\le \sum_{a_1=2}^{\lfloor \frac{m}{k} \rfloor} \binom{m-k-2}{k-2} \Big/ \binom{m}{k} \\
&\le \frac{m-k}{k} \binom{m-k-2}{k-2} \Big/ \left(\frac{m(m-1)}{k(k-1)} \binom{m-2}{k-2} \right) \\
&= \frac{(m-k)(k-1)}{m(m-1)} \frac{(m-k-2)(m-k-3)...(m-2k+1)}{(m-2)(m-3)...(m-k+1)} \\
&\le \frac{(1/4)(m-1)^2}{m(m-1)} \prod_{s=2}^{k-1} \left(\frac{m-s-k}{m-s} \right) \\
&= \frac{m-1}{4m} \prod_{s=2}^{k-1} \left(1 - \frac{k}{m-s} \right)
\end{aligned}$$

using that the maximum of $(m-k)(k-1)$ is $(1/4)(m-1)^2$ when $k = (m+1)/2$. So the first part of (b) is proved. The second part of (b) is obvious. The proof is completed. □

From the theorem, in the worst cases, at least 75% of the parameter sets can be omitted from checking. The example in the next section shows that in the actual computation, there are more parameter sets which can be omitted.

4. An algorithm to finding lower bounds for three–color Ramsey numbers

Based on Theorem 2, we present an algorithm for a lower bound for $R(3,3,q_3)$. The algorithm can be extended to find lower bounds for more general multicolor classical Ramsey numbers with some modifications.

Algorithm 1. *For given $q_1 = q_2 = 3$, $q_3 \ge 3$, perform the following steps:*

(a) Choose a prime number $p = 2m+1$ with primitive root $g = 2$, and positive integers $k_1, k_2, k_3 = m - k_1 - k_2$.

(b) For $i = 1, 2$, do Steps (c) - (g):

(c) Let $M_1 = [0, m-2]$ and $M_2 = [1, m-1] - B_1$.
(d) List all possible subsets B_{i,j_i} in M_i with k_i elements according to the lexicographic order of their elements. If $i = 1$, list all subsets B_{1,j_1} with k_1 elements using the restrictions (2) – (5) of the 2-normalized cyclic graph. For each j_i, do Steps (e) and (f):
(e) Use B_{i,j_i} to form S_{i,j_i}.
(f) Compute the clique number $[S_{i,j_i}]$ of $G_p(S_{i,j_i})$. If $[S_{i,j_i}] < q_i$, then let $S_i = S_{i,j_i}, B_i = B_{i,j_i}$. Go to (b) if $i = 1$, and go to (h) if $i = 2$, then let $i = i + 1$.
(g) Conclude that for the chosen p, k_1, k_2, the method cannot produce a lower bound. Stop.
(h) Let $S_3 = Z_p^+ - \cup_{i=1}^2 S_i$, compute the clique number $[S_3]$. If $[S_3] < q_3$, go to (i). If $[S_3] \geq q_3$, go to (g).
(i) Conclude that $R(3, 3, q_3) \geq p + 1$. Stop.

Note that in practice, for the chosen p, k_1, k_2 the algorithm stops at Step (g) frequently, therefore we need to choose another set of p, k_1, k_2 and do the same steps in the algorithm again. The correct terms p, k_1, k_2 for effective parameter sets are usually obtained after many experiments. In general, we need to select a prime number p, not too small to produce a new lower bound, and not too large so the amount of calculations can be performed in a computer.

We present an example to illustrate our algorithm. The lower bound was initially obtained in [2].

Example 1. $R(3, 3, 4) \geq 30$.

In this case $q_3 = 4$. After experimenting with different choices, we pick $p = 29$ in Step (a) that gives $g = 2$ and $m = 14$, and choose $k_1 = k_2 = 4$, so $k_3 = 6$. There are 34 subsets B_{1,j_1} in $[0, 14 - a_1]$ fitting $2 \leq a_1 \leq 14/4$ with 4 elements:

$\{0,2,4,6\}, \{0,2,4,7\}, \{0,2,4,8\}, \{0,2,4,9\}, \{0,2,4,10\}, \{0,2,4,11\},$
$\{0,2,4,12\}, \{0,2,5,7\}, \{0,2,5,8\}, \{0,2,5,9\}, \{0,2,5,10\}, \{0,2,5,11\},$
$\{0,2,5,12\}, \{0,2,6,8\}, \{0,2,6,9\}, \{0,2,6,10\}, \{0,2,6,11\}, \{0,2,6,12\},$
$\{0,2,7,9\}, \{0,2,7,10\}, \{0,2,7,11\}, \{0,2,7,12\}, \{0,2,8,10\}, \{0,2,8,11\},$
$\{0,2,8,12\}, \{0,2,9,11\}, \{0,2,9,12\}, \{0,2,10,12\}, \{0,3,6,9\}, \{0,3,6,10\},$
$\{0,3,6,11\}, \{0,3,7,10\}, \{0,3,7,11\}, \{0,3,8,11\}.$

After Steps (c) and (d), we obtain the subset $S_1 = \{1, 4, 10, 12\}$ which fits the requirement, i.e. the clique number is $[S_1] = 2 < 3 = q_1$. Using naive computation to find a suitable subset with 4 elements from 14 positions in $\{1, 2, ...14\}$ would require checking $\binom{14}{4} = 1001$ subsets. Using our algorithm, we find three parameter sets:

$S_1 = \{1, 4, 10, 12\}$, $S_2 = \{2, 5, 8, 9\}$, $S_3 = \{3, 6, 7, 11, 13, 14\}$
with three clique numbers respectively $[S_1] = 2$, $[S_2] = 2$, $[S_3] = 3$, therefore $R(3, 3, 4) \geq 30$.

The example shows, in actual computation the real percentage of the parameter sets needing to be checked to obtain S_1 is much less than 25%, which is the conservative estimate from Theorem 3. The percentage in the example is $34/1001 \approx 3.4\%$.

5. Two new lower bounds for Ramsey numbers

Using our algorithm, we have found two sets of effective parameter sets, and these sets yield the new lower bounds in (1) that yield the following theorem:

Theorem 4. $R(3, 3, 12) \geq 182, \quad R(3, 3, 13) \geq 212.$

Proof. For each p, with primitive root $g = 2$, we use Algorithm 1 to obtain the parameter sets S_1, S_2, S_3 and their related clique numbers. We list them as follows:
(a) Let $p = 181$ which gives $g = 2$. When
$S_1 = \{1, 4, 7, 19, 22, 32, 35, 45, 48, 50, 53, 56, 59, 62, 65, 76, 79, 89\}$,
$S_2 = \{2, 3, 9, 10, 17, 24, 25, 31, 38, 39, 46, 57, 61, 68, 72, 73, 80, 87\}$,
we obtain clique numbers $[S_1] = [S_2] = 2$, $[S_3] = 11$ which implies that $R(3, 3, 12) \geq 182$.
(b) Let $p = 211$ which gives $g = 2$. When
$S_1 = \{1, 4, 6, 9, 23, 25, 40, 55, 58, 60, 68, 71, 73, 76, 87, 90, 92, 97, 102, 104\}$,
$S_2 = \{8, 11, 12, 13, 18, 27, 28, 32, 42, 48, 49, 63, 65, 72, 79, 82, 86, 88, 89, 103\}$,
we obtain clique numbers $[S_1] = [S_2] = 2$, $[S_3] = 12$ which implies that $R(3, 3, 13) \geq 212$. $\square$

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REFERENCES

[1] **R.L. Graham, B.L. Rothschild and J.H. Spencer,** *Ramsey Theory,* John Wiley, New York, 1990.
[2] **J.G. Kalbfleisch,** *Chromatic graphs and Ramsey's theory,* Ph.D. thesis, Univ. Waterloo, Jan., 1966.
[3] **H. Luo, W. Su and Y.-Q. Shen,** *New bounds of ten classical Ramsey numbers,* Australas. J. Combin., 24 (2001), 81–90.

[4] **S.P. Radziszowski,** *Small Ramsey numbers,* Electron. J. Combin., 1 (1994), DSI, 30pp, updated on July 15, 2002.
[5] **W. Su, H. Luo and Q. Li,** *Lower bounds for multicolor classical Ramsey numbers* $R(q,q,...,q)$, Sci. China Ser. A, 42 (10) (1999), 1019–1024.
[6] **W. Su, H. Luo and Y.-Q. Shen,** *New lower bounds for classical Ramsey numbers* $R(5,13)$ *and* $R(5,14)$, Appl. Math. Lett., 12 (1999), 121–122.
[7] **W. Su, H. Luo, Z. Zhang and G. Li,** *New lower bounds of fifteen classical Ramsey numbers,* Australas. J. Combin., 19 (1999), 91–909.

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Nove donje granice dvaju multikolornih klasičnih Ramsey brojeva

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Sadržaj

U radu se prezentira algoritam za iznalaženje donjih granica multikolornih klasičnih Ramsey brojeva uz korištenje 2-normaliziranih cikličkih grafova prvog reda. Dobivaju se nove donje granice dvaju multikolornih klasičnih Ramsey brojeva: $R(3,3,12) \geq 182$, $R(3,3,13) \geq 212$.