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φ–conformally flat Lorentzian para–Sasakian manifolds

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Abstract. In this study we consider φ–conformally flat, φ–conharmonically flat and φ–projectively flat Lorentzian para–Sasakian manifolds.

1. Preliminaries

Let (M^n, g), $n = \dim M^n \geq 3$, be a connected semi–Riemannian manifold of class C^∞ and ∇ be its Levi–Civita connection. The Riemannian–Christoffel curvature tensor R, the Weyl conformal curvature tensor C (see [12]), the conharmonic curvature tensor K (see [5]) and the projective curvature tensor P (see [12]) of (M^n, g) are defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1}$$

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ +g(Y,Z)\mathcal{S}X - g(X,Z)\mathcal{S}Y] + \frac{\tau}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{2}$$

$$\begin{aligned} K(X,Y)Z =& R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)\mathcal{S}X - g(X,Z)\mathcal{S}Y], \end{aligned} \tag{3}$$

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\left[g(Y,Z)\mathcal{S}X - g(X,Z)\mathcal{S}Y\right], \tag{4}$$

respectively, where $\mathcal{S}$ is the Ricci operator, defined by $S(X,Y) = g(\mathcal{S}X, Y)$, S is the Ricci tensor, $\tau = tr(S)$ is the scalar curvature and $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of M.

In [6], K. Matsumoto introduced the notion of Lorentzian para–Sasakian (briefly LP–Sasakian) manifold. In [8], the authors defined the

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same notion independently and they obtained many results about this type manifold (see also [7] and [11]).

In § **2**, we define an *LP*–Sasakian manifold and review some formulas which will be used in the later sections. In § **3**, we give the main results of the paper.

2. Lorentzian para–Sasakian manifolds

Let M^n be an n–dimensional differentiable manifold equipped with a triple (φ, ξ, η), where φ is a (1,1)–tensor field, ξ is a vector field, η is a 1–form on M^n such that

$$\eta(\xi) = -1, \quad \varphi^2 X = X + \eta(X)\xi, \tag{5}$$

which implies

$$\eta \circ \varphi = 0, \quad \varphi\xi = 0, \quad rank(\varphi) = n - 1.$$

Then M^n admits a Lorentzian metric g, such that

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{6}$$

and M^n is said to admit a *Lorentzian almost paracontact structure* (φ, ξ, η, g). In this case, we have

$$g(X, \xi) = \eta(X), \tag{7}$$

$$\Phi(X, Y) = g(X, \varphi Y) = g(\varphi X, Y) = \Phi(Y, X),$$

and

$$(\nabla_X \Phi)(Y,Z) = g\big(Y, (\nabla_X \Phi)Z\big) = (\nabla_X \Phi)(Z, Y),$$

where ∇ is the covariant differentiation with respect to g. The Lorentzian metric g makes a timelike unit vector field, that is, $g(\xi, \xi) = -1$. The manifold M^n equipped with a Lorentzian almost paracontact structure (φ, ξ, η, g) is said to be a *Lorentzian almost paracontact manifold* (see [6], [7]).

In (5) if we replace ξ by $-\xi$, then the triple (φ, ξ, η) is an almost paracontact structure on M^n defined by Sato ([9]). The Lorentzian metric given by (6) stands analogous to the almost paracontact Riemannian metric for any almost paracontact manifold (see [9], [10]).

A Lorentzian almost paracontact manifold M^n equipped with the structure (φ, ξ, η, g) is called *Lorentzian paracontact manifold* (see [6]) if

$$\Phi(X, Y) = \frac{1}{2}\big((\nabla_X \eta)Y + (\nabla_Y \eta)X\big).$$

A Lorentzian almost paracontact manifold M^n equipped with the structure (φ, ξ, η, g) is called *Lorentzian para–Sasakian manifold* (briefly *LP*–Sasakian manifold) (see [6]) if

$$(\nabla_X \varphi)Y = g(\varphi X, \varphi Y)\xi + \eta(Y)\varphi^2 X.$$

In an *LP*–Sasakian manifold the 1–form η is closed. Also in [6], it is proved that if an n–dimensional Lorentzian manifold (M^n, g) admits a timelike unit vector field ξ such that the 1–form η associated to ξ is closed and satisfies

$$(\nabla_X \nabla_Y \eta)Z = g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z),$$

then M^n admits an *LP*–Sasakian structure.

Further, on such an *LP*–Sasakian manifold M^n with the structure (φ, η, ξ, g), the following relations hold:

$$R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X, \quad R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{8}$$

and

$$S(X, \xi) = (n-1)\eta(X), \tag{9}$$

$$S(\varphi X, \varphi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \tag{10}$$

for any $X, Y \in \chi(M^n)$ (see [6] and [8]).

An *LP*–Sasakian manifold M^n is said to be η*–Einstein* if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{11}$$

for any vector fields X and Y, where a, b are functions on M^n (see [12] and [3]).

3. Main results

In this section we consider φ–conformally flat, φ–conharmonically flat and φ–projectively flat Lorentzian para–Sasakian manifolds.

Let C be the Weyl conformal curvature tensor of M^n. Since at each point $p \in M^n$ the tangent space $T_p(M^n)$ can be decomposed into the direct sum $T_p(M^n) = \varphi(T_p(M^n)) \oplus L(\xi_p)$, where $L(\xi_p)$ is a 1–dimensional linear subspace of $T_p(M^n)$ generated by ξ_p, we have a map:

$$C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \varphi(T_p(M^n)) \oplus L(\xi_p).$$

It may be natural to consider the following particular cases:

(1) $C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to L(\xi_p)$, that is, the projection of the image of C in $\varphi(T_p(M))$ is zero.

(2) $C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \varphi(T_p(M^n))$, that is, the projection of the image of C in $L(\xi_p)$ is zero.

(3) $C : \varphi(T_p(M^n)) \times \varphi(T_p(M^n)) \times \varphi(T_p(M^n)) \to L(\xi_p)$, that is, when C is restricted to $(T_p(M^n)) \times \varphi(T_p(M^n)) \times \varphi(T_p(M^n))$, the projection of the image of C in $\varphi(T_p(M^n))$ is zero. This condition is equivalent to

$$\varphi^2 C(\varphi X, \varphi Y)\varphi Z = 0, \tag{12}$$

(see [4]).

Definition 1. A differentiable manifold (M^n, g), $n > 3$, satisfying the condition (12) is called φ–conformally flat.

The cases (1) and (2) were considered in [13] and [14] respectively. The case (3) was considered in [4] for the case M is a K–contact manifold.

Furthermore in [1], the authors studied (k, μ)–contact metric manifolds satisfying (12). Now our aim is to find the characterization of LP–Sasakian manifolds satisfying the condition (12).

Theorem 2. *Let M^n be an n–dimensional, ($n > 3$), φ–conformally flat LP–Sasakian manifold. Then M^n is an η–Einstein manifold.*

Proof. Suppose that (M^n, g), $n > 3$, is a φ–conformally flat LP–Sasakian manifold. It is easy to see that $\varphi^2 C(\varphi X, \ \varphi Y)\varphi Z = 0$ holds if and only if

$$g(C(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0,$$

for any $X, Y, Z, W \in \chi(M^n)$. So by the use of (2) φ –conformally flat means

$$g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) = \frac{1}{n-2}\,[g(\varphi Y, \varphi Z)S(\varphi X, \varphi W)$$
$$-g(\varphi X, \varphi Z)S(\varphi Y, \varphi W) + g(\varphi X, \varphi W)S(\varphi Y, \varphi Z) - g(\varphi Y, \varphi W)S(\varphi X, \varphi Z)]$$
$$-\frac{\tau}{(n-1)(n-2)}\,[g(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - g(\varphi X, \varphi Z)g(\varphi Y, \varphi W)]. \tag{13}$$

Let $\{e_1, ..., e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M^n. Using that $\{\varphi e_1,...,\varphi e_{n-1},\xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (13) and sum up with respect to i, then

$$\sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) = \frac{1}{n-2}\sum_{i=1}^{n-1}[g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i)$$
$$-g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i) + g(\varphi e_i, \varphi e_i)S(\varphi Y, \varphi Z) - g(\varphi Y, \varphi e_i)S(\varphi e_i, \varphi Z)]$$
$$-\frac{\tau}{(n-1)(n-2)}\sum_{i=1}^{n-1}[g(\varphi Y, \varphi Z)g(\varphi e_i, \varphi e_i) - g(\varphi e_i, \varphi Z)g(\varphi Y, \varphi e_i)]. \tag{14}$$

It can be easily verify that

$$\sum_{i=1}^{n-1} g(R(\varphi e_i,\varphi Y)\varphi Z,\varphi e_i) = S(\varphi Y,\varphi Z) + g(\varphi Y,\varphi Z), \tag{15}$$

$$\sum_{i=1}^{n-1} S(\varphi e_i,\varphi e_i) = \tau + n - 1, \tag{16}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i,\varphi Z)S(\varphi Y,\varphi e_i) = S(\varphi Y,\varphi Z), \tag{17}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i,\varphi e_i) = n + 1, \tag{18}$$

and

$$\sum_{i=1}^{n-1} g(\varphi e_i,\varphi Z)g(\varphi Y,\varphi e_i) = g(\varphi Y,\varphi Z). \tag{19}$$

So by virtue of (15)–(19) the equation (14) can be written as

$$S(\varphi Y,\varphi Z) = \left(\frac{\tau}{n-1} - 1\right) g(\varphi Y,\varphi Z). \tag{20}$$

Then by making use of (6) and (10), the equation (20) takes the form

$$S(Y,Z) = \left(\frac{\tau}{n-1} - 1\right) g(Y,Z) + \left(\frac{\tau}{n-1} - n\right) \eta(Y)\eta(Z), \tag{21}$$

which implies, from (11), M^n is an η–Einstein manifold. This completes the proof of the theorem.

Definition 3. A differentiable manifold (M^n, g), $n > 3$, satisfying the condition

$$\varphi^2 K(\varphi X,\varphi Y)\varphi Z = 0 \tag{22}$$

is called φ–conharmonically flat. In [2], the authors considered (k,μ)–contact manifolds satisfying (22). Now we will study the condition (22) on LP–Sasakian manifolds.

Theorem 4. *Let M^n be an n–dimensional, ($n > 3$), φ–conharmonically flat LP–Sasakian manifold. Then M^n is an η–Einstein manifold with the zero scalar curvature.*

Proof. Assume that (M^n, g), $n > 3$, is a φ–conformally flat LP–Sasakian manifold. It can be easily seen that $\varphi^2 K(\varphi X, \varphi Y)\varphi Z = 0$ holds if and only if

$$g(K(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0,$$

for any $X, Y, Z, W \in \chi(M^n)$. Using (3) φ–conharmonically flat means

$$g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) = \frac{1}{n-2}[g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) - g(\varphi X, \varphi Z)S(\varphi Y, \varphi W) + g(\varphi X, \varphi W)S(\varphi Y, \varphi Z) - g(\varphi Y, \varphi W)S(\varphi X, \varphi Z)]. \tag{23}$$

Similar to the proof of Theorem 2, we can suppose that $\{e_1, \ldots, e_{n-1}, \xi\}$ is a local orthonormal basis of vector fields in M^n. By using the fact that $\{\varphi e_1, \ldots, \varphi e_{2n}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (23) and sum up with respect to i, then

$$\sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i) = \frac{1}{n-2}\sum_{i=1}^{n-1}[g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) - g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i) + g(\varphi e_i, \varphi e_i)S(\varphi Y, \varphi Z) - g(\varphi Y, \varphi e_i)S(\varphi e_i, \varphi Z)]. \tag{24}$$

So by the use of (15)-(18) the equation (24) turns into

$$-S(\varphi Y, \varphi Z) = (\tau + 1)g(\varphi Y, \varphi Z). \tag{25}$$

Thus applying (6) and (10) into (25) we get

$$S(Y, Z) = -(\tau + 1)\, g(Y, Z) - (n + \tau)\eta(Y)\eta(Z), \tag{26}$$

which gives us, from (11), M^n is an η-Einstein manifold. Hence contracting (26) we obtain $n\tau = 0$, which implies the scalar curvature $\tau = 0$. Our theorem is thus proved.

Similar to Definition 1 and Definition 3 we can state the following:

Definition 5. A differentiable manifold (M^n, g), $n > 3$, satisfying the condition

$$\varphi^2 P(\varphi X, \varphi Y)\varphi Z = 0 \tag{27}$$

is called φ-projectively flat.

Theorem 6. *Let M^n be an n–dimensional, ($n > 3$), φ–projectively flat LP–Sasakian manifold. Then M^n is an Einstein manifold with the scalar curvature $\tau = n(n-1)$.*

Proof. It can be easily seen that $\varphi^2 P(\varphi X\ ,\ \varphi Y)\varphi Z = 0$ holds if and only if

$$g(P(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0,$$

for any $X, Y, Z, W \in \chi(M^n)$. Using (1) and (4) φ–projectively flat means

$$g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) = \frac{1}{n-2}[g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) - g(\varphi X, \varphi Z)S(\varphi Y, \varphi W). \tag{28}$$

In a manner similar to the method in the proof of Theorem 2, choosing $\{e_1, \dots, e_{n-1}, \xi\}$ as a local orthonormal basis of vector fields in M^n and using the fact that $\{\varphi e_1, \dots, \varphi e_{2n}, \xi\}$ is also a local orthonormal basis, putting $X = W = e_i$ in (28) and summing up with respect to i, then we have

$$\sum_{i=1}^{n-1} g(R(\varphi e_i, \varphi Y)\varphi Z, \varphi e_i)$$
$$= \frac{1}{n-2}\sum_{i=1}^{n-1}[g(\varphi Y, \varphi Z)S(\varphi e_i, \varphi e_i) - g(\varphi e_i, \varphi Z)S(\varphi Y, \varphi e_i)]. \tag{29}$$

So applying (15)–(17) into (29) we get

$$nS(\varphi Y, \varphi Z) = \tau g(\varphi Y, \varphi Z).$$

Hence by virtue of (6) and (10) we obtain

$$S(Y, Z) = \frac{\tau}{n} g(Y, Z) + \left(\frac{\tau}{n} - (n-1)\right)\eta(Y)\eta(Z). \tag{30}$$

Therefore from (30), by contraction, we obtain

$$\tau = n(n-1). \tag{31}$$

So substituting (31) into (30) we find

$$S(Y, Z) = (n-1)g(Y, Z),$$

which implies M^n is an Einstein manifold with the scalar curvature $\tau = n(n-1)$. This completes the proof of the theorem.

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φ–konformalno ravne Lorentzian para–Sasakian mnogostrukosti

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Sadržaj

U radu se razmatraju φ–konformalno ravne, φ–konharmonično ravne i φ–projektivno ravne Lorentzian para–Sasakian mnogostrukosti.