Results on the error function and the neutrix convolution

B. Fisher (England), M. Telci and E. Özçağ (Turkey)

Abstract. Some neutrix convolutions of the error function $\text{erf}(x)$ and its associated functions $\text{erf}_+(x)$ and $\text{erf}_-(x)$ with $x^r$, $x^r_+$ and $x^r_-$ are evaluated. Further neutrix convolutions are deduced.

The error function $\text{erf}(x)$, see for example Sneddon [6], is the locally summable function defined by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2) \, du.$$  \hfill (1)

More generally $\text{erf}(\lambda x)$ was defined in [4] for $\lambda \neq 0$ by

$$\text{erf}(\lambda x) = \frac{2}{\sqrt{\pi}} \int_0^{\lambda x} \exp(-u^2) \, du.$$  \hfill (2)

It is easily seen that $\text{erf}(x)$ is an odd function of $x$.

The locally summable functions $\text{erf}_+(\lambda x)$ and $\text{erf}_-(\lambda x)$ were defined for $\lambda \neq 0$ by

$$\text{erf}_+(\lambda x) = H(x)\text{erf}(\lambda x) \quad \text{and} \quad \text{erf}_-(\lambda x) = H(-x)\text{erf}(\lambda x),$$

where $H$ denotes Heaviside’s function. Note that

$$\text{erf}_+[\lambda(-x)] = -\text{erf}_-(\lambda x), \quad \text{and} \quad \text{erf}_-[-\lambda x] = -\text{erf}_+(\lambda x).$$  \hfill (3)

If $f$ and $g$ are locally summable functions then the classical convolution $f * g$ of $f$ and $g$ is given by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t) \, dt = \int_{-\infty}^{\infty} f(x-t)g(t) \, dt.$$  

1 This research was supported by TUBITAK.

2000 Mathematics Subject Classification: 46F10.

Key words and phrases: Error function, distribution, neutrix, neutrix limit, convolution, neutrix convolution.
for all values of \( x \) for which the integral exists. It follows easily that if \( f \ast g \) exists, then

\[
f \ast g = g \ast f, \\
(f \ast g)' = f' \ast g = f \ast g'.
\]

Before proving our results on the convolution, we need the following easily proved lemma:

**Lemma 1.**

\[
\begin{align*}
\alpha_{2r}(x) &= \int_0^x t^{2r} \exp(-t^2) \, dt \\
&= -\sum_{i=0}^{r-1} \frac{(2r)!}{2^{2i+1}r!(2r-2i)!} x^{2r-2i-1} \exp(-x^2) + \frac{(2r)!}{2^{2r+1}r!} \exp(x), \\
\alpha_{2r+1}(x) &= \int_0^x t^{2r+1} \exp(-t^2) \, dt \\
&= -\sum_{i=0}^{r} \frac{r!}{2(r-i)!} x^{2r-2i} \exp(-x^2) + \frac{r!}{2}
\end{align*}
\]

for \( r = 0, 1, 2, \ldots \), where the sum in (4) is empty when \( r = 0 \).

We now let \( \mathcal{D} \) be the space of infinitely differentiable functions with compact support and let \( \mathcal{D}' \) be the space of distributions defined on \( \mathcal{D} \).

**Definition 1.** The *convolution* \( f \ast g \) of two distributions \( f \) and \( g \) in \( \mathcal{D}' \) is defined by the equation

\[
\langle (f \ast g)(x), \varphi \rangle = \langle f(y), \langle g(x), \varphi(x+y) \rangle \rangle
\]

for arbitrary \( \varphi \) in \( \mathcal{D} \), provided \( f \) and \( g \) satisfy either of the conditions

(a) either \( f \) or \( g \) has bounded support,

(b) the supports of \( f \) and \( g \) are bounded on the same side,

see Gel’fand and Shilov [5].

Note that if \( f \) and \( g \) are locally summable functions satisfying either of the above conditions and the classical convolution \( f \ast g \) exists, then it is in agreement with Definition 1.

This definition of the convolution is rather restrictive and so the non–commutative neutrix convolution was introduced in [2]. In order to define the neutrix convolution product we first of all let \( \tau \) be a function in \( \mathcal{D} \) satisfying the following properties:
(i) $\tau(x) = \tau(-x)$,
(ii) $0 \leq \tau(x) \leq 1$,
(iii) $\tau(x) = 1$ for $|x| \leq \frac{1}{2}$,
(iv) $\tau(x) = 0$ for $|x| \geq 1$.

The function $\tau_n$ is now defined by

$$\tau_n(x) = \begin{cases} 
1, & |x| \leq n, \\
\tau(n^n x - n^{n+1}), & x > n, \\
\tau(n^n x + n^{n+1}), & x < -n,
\end{cases}$$

for $n = 1, 2, \ldots$.

**Definition 2.** Let $f$ and $g$ be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the neutrix convolution $f \odot g$ is defined as the neutrix limit of the sequence $\{f_n \ast g\}$, provided that the limit $h$ exists in the sense that

$$\lim_{n \to \infty} \langle f_n \ast g, \phi \rangle = \langle h, \phi \rangle,$$

for all $\phi$ in $\mathcal{D}$, where $N$ is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range $N''$ the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \ln^r n \quad (\lambda > 0, r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as $n$ tends to infinity.

Note that in this definition the convolution $f_n \ast g$ is defined in Gel’fand and Shilov’s sense, the distribution $f_n$ having bounded support. Note also that because of the lack of symmetry in the definition of $f \odot g$, the neutrix convolution is in general non-commutative.

The following theorem was proved in [2], showing that the neutrix convolution is a generalization of the convolution.

**Theorem 1.** Let $f$ and $g$ be distributions in $\mathcal{D}'$ satisfying either condition (a) or condition (b) of Gel’fand and Shilov’s definition. Then the neutrix convolution $f \odot g$ exists and

$$f \odot g = f \ast g.$$

The next theorem was also proved in [2].
**Theorem 2.** Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) and suppose that \( f \ast g \) exists, then the neutrix convolution \( f \ast g' \) exists and
\[
(f \ast g)' = f \ast g'.
\] (6)

Note however that \( (f \ast g)' \) is not necessarily equal to \( f' \ast g \). We do however have the following lemma which was proved in [3].

**Lemma 2.** Let \( f \) and \( g \) be distributions in \( \mathcal{D}' \) and suppose that \( f \ast g \) exists. If \( \lim_{n \to \infty} \langle f(\tau_n) \ast g, \varphi \rangle \) exists and equals \( \langle h, \varphi \rangle \) for all \( \varphi \) in \( \mathcal{D}' \), then the neutrix convolution \( f' \ast g \) exists and
\[
(f \ast g)' = f' \ast g + h.
\] (7)

In order to define further neutrix convolution products, we increase our set of negligible functions given in Definition 2 to also include finite linear sums of the functions
\[
n^r \text{erf}(\lambda n), \quad r = 1, 2, \ldots, \quad \lambda \neq 0.
\]

The following theorem was proved in [4].

**Theorem 3.** The function \( n^r \text{erf}[\lambda (\alpha \pm n)], \varphi \) is a negligible function for \( r = 1, 2, \ldots, \lambda \neq 0 \) and all \( \varphi \) in \( \mathcal{D} \).

We now prove

**Theorem 4.** If \( \lambda \neq 0 \), then the neutrix convolution \( \text{erf}(\lambda x) \ast x_+^r \) exists and
\[
\text{erf}(\lambda x) \ast x_+^r = \frac{2}{(r + 1) \sqrt{\pi}} \sum_{i=0}^{r+1} \binom{r+1}{i} (-\lambda)^{-i} \alpha_i(\lambda x) x^{r-i+1} + \\
+ \frac{2}{(r + 1) \sqrt{\pi}} \sum_{i=1}^{r+1} \binom{r+1}{i} (-\lambda)^{-i} \beta_i x^{r-i+1}
\] (8)
for \( r = 0, 1, 2, \ldots \).

**Proof.** We put \( \text{erf}_n(\lambda x) = \text{erf}(\lambda x) \tau_n(x) \) for \( n = 1, 2, \ldots \). Since \( \text{erf}_n(\lambda x) \) has compact support, the classical convolution \( \text{erf}_n(\lambda x) \ast x_+^r \) exists and
\[
\frac{\sqrt{\pi}}{2} \text{erf}_n(\lambda x) \ast x_+^r = \int_{x-n}^{x} (x - t)^r \text{erf}(\lambda t) \, dt + \\
\int_{-x-n}^{-x} (x - t)^r \tau_n(t) \text{erf}(\lambda t) \, dt \\
= I_1 + I_2.
\] (9)
It is easily seen that
\[
\lim_{n \to \infty} I_2 = 0. \tag{10}
\]

Further,
\[
I_1 = \int_{-\infty}^{\lambda x} (x - t)^r \int_0^{\lambda t} \exp(-u^2) \, du \, dt
\]
\[
= \int_0^{\lambda x} \exp(-u^2) \left( \int_{\lambda t}^{\lambda x} (x - t)^r \, dt + \int_{-\infty}^{0} (x - t)^r \, dt + \int_{-\lambda n}^{0} (x - t)^r \, dt \right) \, du
\]
\[
= \frac{1}{r+1} \int_0^{\lambda x} (x - u/\lambda)^{r+1} \exp(-u^2) \, du + \frac{1}{r+1} \int_{-\lambda n}^{0} [(x - u/\lambda)^{r+1} - (x + n)^{r+1}] \exp(-u^2) \, du
\]
\[
= \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-\lambda)^{-i} \alpha_i(\lambda x)x^{-i+1} + \frac{1}{r+1} \sum_{i=1}^{r+1} \binom{r+1}{i} (-\lambda)^{-i} \alpha_i(-\lambda n)x^{-i+1}
\]
\[
+ \frac{\sqrt{\pi}}{2(r+1)} \sum_{i=1}^{r+1} \binom{r+1}{i} x^{-i+1} n \operatorname{erf}(\lambda n).
\]

Noting that
\[
\lim_{n \to \infty} \alpha_{2i}(-\lambda n) = \frac{(2i)! \sqrt{\pi} \sgn \lambda}{2^{2i+1} i!}, \quad \lim_{n \to \infty} \alpha_{2i+1}(-\lambda n) = \frac{i!}{2}
\]
for \(i = 0, 1, 2, \ldots\) and using Theorem 3, it follows that
\[
N - \lim_{n \to \infty} I_1 = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-\lambda)^{-i} \alpha_i(\lambda x)x^{-i+1} + \frac{1}{r+1} \sum_{i=1}^{r+1} \binom{r+1}{i} (-\lambda)^{-i} \beta_i x^{-i+1}.
\]
\[
\tag{11}
\]
Equation (8) now follows from equations (9), (10) and (11).

**Corollary 4.1.** If \(\lambda \neq 0\), then the neutrix convolution \(\text{erf}(\lambda x) \odot x^r\) exists and
\[
\text{erf}(\lambda x) \odot x^r = \frac{2(-1)^r}{(r+1)\sqrt{\pi}} \sum_{i=0}^{r+1} \binom{r+1}{i} \lambda^{-i} (-1)^{i+1} \alpha_i(\lambda x)x^{-i+1} + \frac{2(-1)^r}{(r+1)\sqrt{\pi}} \sum_{i=1}^{r+1} \binom{r+1}{i} \lambda^{-i} \beta_i x^{-i+1}
\]
\[
for r = 0, 1, 2, \ldots
\]

for \(r = 0, 1, 2, \ldots\).
Proof. It follows from equations (4) and (5) that
\[ \alpha_i(-x) = (-1)^{i+1} \alpha_i(x). \]
Equation (12) now follows on replacing \( x \) by \( -x \) in equation (8).

Corollary 4.2. If \( \lambda \neq 0 \), then the neutrix convolutions \( \exp(-\lambda^2 x^2) \ast x^r_+ \) and \( \exp(-\lambda^2 x^2) \ast x^r_- \) exist and

\[
\exp(-\lambda^2 x^2) \ast x^r_+ = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-\lambda)^{-i} \alpha'_i(\lambda x)x^{r-i+1} + \]
\[
- \frac{1}{r+1} \sum_{i=1}^{r} \binom{r+1}{i+1} (i+1)(-\lambda)^{-i-1} \alpha_i(\lambda x)x^{r-i+1} + \]
\[
- \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i+1} (i+1)(-\lambda)^{-i-1} \beta_i x^{r-i},
\]
\[
\exp(-\lambda^2 x^2) \ast x^r_- = \frac{(-1)^{r+1}}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i+1} (-\lambda)^{-i} \alpha'_i(\lambda x)x^{r-i+1} + \]
\[
- \frac{(-1)^r}{r+1} \sum_{i=1}^{r} \binom{r+1}{i+1} (i+1)(-1)^i \lambda^{-i-1} \alpha_i(\lambda x)x^{r-i} + \]
\[
+ \frac{(-1)^r}{r+1} \sum_{i=0}^{r} \binom{r+1}{i+1} (i+1)\lambda^{-i-1} \beta_i x^{r-i}
\]
for \( r = 0, 1, 2, \ldots \).

Proof. With \( x < n + n^{-n} \), we have

\[
[\text{erf}(\lambda x) \tau_n'(x)] \ast x^r_+ = \int_{-n-n^{-n}}^{-n} (x-t)^r \text{erf}(\lambda t) \, d\tau_n(t)
\]
\[
= (x+n)^r \text{erf}(-\lambda n) - \frac{2}{\sqrt{\pi}} \int_{-n-n^{-n}}^{-n} (x-t)^r \exp(-\lambda^2 t^2) \tau_n(t) \, dt + \]
\[
+ r \int_{-n-n^{-n}}^{-n} (x-t)^{r-1} \text{erf}(\lambda t) \tau_n(t) \, dt
\]
\[
= I_1 + I_2 + I_3.
\]
It is clear that
\[ N \lim_{n \to \infty} I_1 = \lim_{n \to \infty} x^r \text{erf}(-\lambda n) = -\text{sgn} \lambda x^r, \]  
and it follows from equations (15), (16) and (17) that
\[ N \lim_{n \to \infty} [\text{erf}(\lambda x) r'_n(x)] * x^r = -\text{sgn} \lambda x^r. \] (18)

Differentiating equation (8) partially with respect to \( x \) and using Lemma 2, Theorem 3 and equation (18) gives equation (13). Equation (14) follows similarly from equation (12).

**Corollary 4.3** If \( \lambda \neq 0 \), then the neutrix convolutions \([x \exp(-\lambda^2x^2)] \odot x^r_+\) and \([x \exp(-\lambda^2x^2)] \odot x^r_-\) exist and
\[
[x \exp(-\lambda^2x^2)] \odot x^r_+ = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-\lambda)^{-i} \alpha_i'(\lambda x) x^{r-i+2} + \\
+ \frac{1}{r+1} \sum_{i=1}^{r+1} \binom{r+1}{i} (-\lambda)^{-i-1} \alpha_i(\lambda x) x^{r-i+1} + \\
+ \frac{1}{r+1} \sum_{i=1}^{r+1} \binom{r+1}{i} i(-\lambda)^{-i-1} \beta_i x^{r-i+1}, \tag{19}
\]
\[
[x \exp(-\lambda^2x^2)] \odot x^r_- = \frac{(-1)^r}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-\lambda)^{-i} \alpha_i'(\lambda x) x^{r-i+2} + \\
- \frac{(-1)^r}{r+1} \sum_{i=1}^{r+1} \binom{r+1}{i} (-\lambda)^{-i-1} \alpha_i(\lambda x) x^{r-i+1} + \\
- \frac{(-1)^r}{r+1} \sum_{i=1}^{r+1} \binom{r+1}{i} i\lambda^{-i-1} \beta_i x^{r-i+1}, \tag{20}
\]
for \( r = 0, 1, 2, \ldots \).

**Proof.** Equation (19) follows on differentiating equation (8) partially with respect to \( x \). Equation (20) follows similarly from equation (12).

**Corollary 4.4** If \( \lambda \neq 0 \), then the neutrix convolution \( \text{erf}(\lambda x) \odot x^r \) exists and
\[
\text{erf}(\lambda x) \odot x^r = -\text{sgn} \lambda x^{r} \sum_{i=0}^{\left\lceil \frac{r+1}{2} \right\rceil} \frac{(2i)!}{(2^i-1)!} \lambda^{-2i} x^{r-2i+1} \tag{21}
\]
for $r = 0, 1, 2, \ldots$, where $[a]$ denotes the integer part of $a$.

**Proof.** Noting that $x^r = x^r_+ + (-1)^r x^r_-$, it follows from equations (8) and (12) that

$$\text{erf}(\lambda x) \otimes x^r = \frac{2}{(r+1)\sqrt{\pi}} \sum_{i=0}^{r+1} \binom{r+1}{i} [1 + (-1)^i\lambda^{-i}\beta_i] x^{r-i+1}$$

$$= \frac{4}{(r+1)\sqrt{\pi}} \sum_{i=0}^{[(r+1)/2]} \binom{r+1}{2i} \lambda^{-2i}\beta_{2i} x^{r-2i+1}$$

and equation (21) follows.

**Theorem 5.** If $\lambda \neq 0$, then the neutrix convolution $[x^s \exp(-\lambda^2 x^2)] \otimes x^r$ exists and

$$[x^s \exp(-\lambda^2 x^2)] \otimes x^r = \frac{\sqrt{\pi} \text{sgn} \lambda}{r+1} \sum_{i=0}^{[(r+1)/2]} \binom{r+1}{2i} \frac{(2i+1)(2s+2i)!}{2^{2s+2i}(s+i)!} \lambda^{-2s-2i} x^{r-2i}, \quad (22)$$

$$[x^{s+1} \exp(-\lambda^2 x^2)] \otimes x^r = \frac{\sqrt{\pi} \text{sgn} \lambda}{r+1} \sum_{i=0}^{[(r+1)/2]} \binom{r+1}{2i} \frac{i(2s+2i)!}{2^{2s+2i-1}(s+i)!} \lambda^{-2s-2i} x^{r-2i+1}, \quad (23)$$

for $r, s = 0, 1, 2, \ldots$.

**Proof.** With $x < n + n^{-n}$, we have

$$[t^s \exp(-\lambda^2 x^2) \tau^*_n(x)] \otimes x^r_+ = \int_{-n-n^{-n}}^{-n} (x - t)^r t^s \exp(-\lambda^2 t^2) \, d\tau_n(t)$$

$$= (x + n)^r (-n)^s \exp(-\lambda^2 n^2)$$

$$- \int_{-n-n^{-n}}^{-n} [(x - t)^r t^s \exp(-\lambda^2 t^2)]' \tau_n(t) \, dt$$

and it follows easily that

$$\lim_{n \to \infty} [t^s \exp(-\lambda^2 x^2) \tau^*_n(x)] \otimes x^r_+ = 0. \quad (24)$$

Differentiating equation (21) partially with respect to $x$ and using the Lemma and equation (24) now gives equation (22) for the case $s = 0$. 
Now suppose that equation (22) holds for some $s$. Then differentiating equation (22) partially with respect to $\lambda$ we get

$$ [x^{2s+2} \exp(-\lambda^2 x^2)] \ast x^r = $$

$$ = -\sqrt{\pi} \frac{\text{sgn} \lambda}{r+1} \sum_{i=0}^{(r+1)/2} \binom{r+1}{2i} \frac{(2i+1)(2i+2s)!}{2^{2s+2i+1}(s+i)!} (2s+2i+1) \lambda^{-2s-2i-3} x^{r-2i} $$

and equation (22) follows by induction.

Next, differentiating equation (21) partially with respect to $\lambda$ gives equation (23) for the case $s = 0$.

Now suppose that equation (23) holds for some $s$. Then differentiating equation (23) partially with respect to $\lambda$ we get

$$ [x^{2s+3} \exp(-\lambda^2 x^2)] \ast x^r = $$

$$ = \sqrt{\pi} \frac{\text{sgn} \lambda}{r+1} \sum_{i=0}^{(r+1)/2} \binom{r+1}{2i} \frac{i(2s+2i)!}{2^{2s+2i}(s+i)!} (2s+2i+1) \lambda^{-2s-2i-3} x^{r-2i+1} $$

and equation (23) follows by induction.

For further related results, see [4].

REFERENCES

Rezultati o funkciji greške i neutrix konvoluciji

B. Fisher, M. Telci i E. Özcag

Sadržaj

U radu se izračunavaju neke neutrix konvolucije funkcije greške erf(x) i njenih pridruženih funkcija erf+(x) i erf−(x) sa x', x'+ i x'−. Daljnje neutrix konvolucije su izvedene.