B.Y. Chen inequalities for slant submanifolds in generalized complex space forms

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Abstract. In this article, we establish B.Y. Chen inequalities for slant submanifolds \( M \) in generalized complex space forms \( \tilde{M}(c, \alpha) \) (\( RK \)-manifolds of constant holomorphic sectional curvature \( c \) and of constant type \( \alpha \)).

1. Preliminaries

In the introduction of [3], B.Y. Chen recalls as one of the basic problems in submanifold theory:

“Find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold”.

In [2], B.Y. Chen proved the Chen inequality for submanifolds in real space forms. In an analogous way, we established Chen inequalities for \( \theta \)-slant submanifolds in complex space forms \( \tilde{M}(c) \) [9].

Let \( \tilde{M} \) be an almost Hermitian manifold with almost complex structure \( J \) and Riemannian metric \( g \). One denotes by \( \tilde{\nabla} \) the operator of covariant differentiation with respect to \( g \) on \( \tilde{M} \).

Definition. If the almost complex structure \( J \) satisfies

\[
(\tilde{\nabla}_X J)Y + (\tilde{\nabla}_Y J)X = 0,
\]

for any vector fields \( X \) and \( Y \) on \( \tilde{M} \), then the manifold \( \tilde{M} \) is called a nearly-Kaehler manifold [11].

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Remark. The above condition is equivalent to
\[(\bar{\nabla}_X J)X = 0, \quad \forall X \in \Gamma T\bar{M}.\]

For an almost complex structure $J$ on the manifold $M$, the Nijenhuis tensor field is defined by
\[N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],\]
for any vector fields $X, Y$ tangent to $M$, where $[,]$ is the Lie bracket.

A necessary and sufficient condition for a nearly–Kaehler manifold to be Kaehler is the vanishing of the Nijenhuis tensor $N_J$. Any 4-dimensional nearly–Kaehler manifold is a Kaehler manifold.

Example. Let $S^6$ be the 6-dimensional unit sphere defined as follows. Let $E^7$ be the set of all purely imaginary Cayley numbers. Then $E^7$ is a 7-dimensional subspace of the Cayley algebra $C$. Let $\{1 = e_0, e_1, \ldots, e_6\}$ be a basis of the Cayley algebra, 1 being the unit element of $C$. If $X = \sum_{i=0}^{6} x^i e_i$ and $Y = \sum_{i=0}^{6} y^i e_i$ are two elements of $E^7$, one defines the scalar product in $E^7$ by
\[< X, Y > = \sum_{i=0}^{6} x^i y^i,\]
and the vector product by
\[X \times Y = \sum_{i \neq j} x^i y^j e_i \ast e_j,\]
$\ast$ being the multiplication operation of $C$.

Consider the 6-dimensional unit sphere $S^6$ in $E^7$:
\[S^6 = \{X \in E^7 \mid < X, X >= 1\}.\]

The scalar product in $E^7$ induces the natural metric tensor field $g$ on $S^6$.

The tangent space $T_X S^6$ at $X \in S^6$ can naturally be identified with the subspace of $E^7$ orthogonal to $X$.

Define the endomorphism $J_X$ on $T_X S^6$ by
\[J_X Y = X \times Y, \quad \text{for } Y \in T_X S^6.\]

It is easy to see that
\[g(J_X Y, J_X Z) = g(Y, Z), \quad Y, Z \in T_X S^6.\]
The correspondence \( X \mapsto JX \) defines a tensor field \( J \) such that \( J^2 = -I \).

Consequently, \( S^6 \) admits an almost Hermitian structure \( (J, g) \).

This structure is a non–Kählerian nearly–Kählerian structure (its Betti numbers of even order are 0).

We will consider a class of almost Hermitian manifolds, called \( RK \)-manifolds, which contains the nearly–Kähler manifolds.

**Definition** [10]. A \( RK \)-manifold \((\tilde{M}, J, g)\) is an almost Hermitian manifold for which the curvature tensor \( \tilde{R} \) is invariant by \( J \), i.e.

\[
\tilde{R}(JX, JY, JZ, JW) = \tilde{R}(X, Y, Z, W),
\]

for any \( X, Y, Z, W \in \Gamma T\tilde{M} \), where \( \Gamma T\tilde{M} \) denotes the set of sections of the tangent bundle \( T\tilde{M} \).

An almost Hermitian manifold \( \tilde{M} \) is of pointwise constant type if for any \( p \in \tilde{M} \) and \( X \in T_p\tilde{M} \) we have \( \lambda(X, Y) = \lambda(X, Z) \), where

\[
\lambda(X, Y) = \tilde{R}(X, Y, JX, JY) - \tilde{R}(X, Y, X, Y)
\]

and \( Y \) and \( Z \) are unit tangent vectors on \( \tilde{M} \) at \( p \), orthogonal to \( X \) and \( JX \), i.e. \( g(X, X) = g(Y, Y) = 1 \), \( g(X, Y) = g(JX, Y) = g(X, Z) = g(JX, Z) = 0 \).

The manifold \( \tilde{M} \) is said to be of constant type if \( \lambda(X, Y) \) is a constant function for any unit \( X, Y \in \Gamma T\tilde{M} \) with \( g(X, Y) = g(JX, Y) = 0 \).

Recall the following result [10].

**Theorem.** Let \( \tilde{M} \) be a \( RK \)-manifold. Then \( \tilde{M} \) is of pointwise constant type if and only if there exists a function \( \alpha \) on \( \tilde{M} \) such that

\[
\lambda(X, Y) = \alpha[g(X, X)g(Y, Y) - (g(X, Y))^2 - (g(X, JY))^2],
\]

for any \( X, Y \in \Gamma T\tilde{M} \).

Moreover, \( \tilde{M} \) is of constant type if and only if the above equality holds for a constant \( \alpha \).

In this case, \( \alpha \) is the constant type of \( \tilde{M} \).

**Definition.** A generalized complex space form is a \( RK \)-manifold of constant holomorphic sectional curvature and of constant type.

We will denote a generalized complex space form by \( \tilde{M}(c, \alpha) \), where \( c \) is the constant holomorphic sectional curvature and \( \alpha \) the constant type, respectively.
Each complex space form is a generalized complex space form. The converse statement is not true. The sphere $S^6$ endowed with the standard nearly-Kaehler structure is an example of generalized complex space form which is not a complex space form.

Let $\tilde{M}(c, \alpha)$ be a generalized complex space form of constant holomorphic sectional curvature $c$ and of constant type $\alpha$. Then the curvature tensor $\tilde{R}$ of $\tilde{M}(c, \alpha)$ has the following form [10]:

$$\tilde{R}(X, Y)Z = \frac{c + 3\alpha}{4} [g(Y, Z)X - g(X, Z)Y] + \frac{c - \alpha}{4} [g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ].$$

(1.1)

Let $M$ be an $n$–dimensional submanifold of a $2m$–dimensional generalized complex space form $\tilde{M}(c, \alpha)$. We denote by $K(\pi)$ the sectional curvature of $M$ associated with a plane section $\pi \subset T_pM, p \in M$. Let $\nabla$ and $h$ be the Levi–Civita connection of $M$ and the second fundamental form, respectively. Then the equation of Gauss is given by

$$\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),$$

(1.2)

for any vectors $X, Y, Z, W$ tangent to $M$, where $R$ is the Riemann curvature tensor of $M$.

We denote by $H$ the mean curvature vector at $p \in M$, i.e.

$$H(p) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i),$$

(1.3)

where $\{e_1, \ldots, e_{2m}\}$ is an orthonormal basis of the tangent space $T_p\tilde{M}(c, \alpha)$, such that $\{e_1, \ldots, e_n\}$ are tangent to $M$.

Suppose $L$ is a $k$–plane section of $T_pM$ and $\{e_1, \ldots, e_k\}$ an orthonormal basis of $L$.

The scalar curvature $\tau$ of the $k$–plane section $L$ is given by

$$\tau(L) = \sum_{1 \leq i < j \leq k} K_{ij}.$$  

(1.4)

The Riemannian invariants of a Riemannian manifold are the intrinsic characteristics of the Riemannian manifold. We recall [4] that the Chen invariants are a string of Riemannian invariants on a Riemannian manifold, which are known as Chen invariants.
The Chen first invariant of a Riemannian manifold $M$ is given by
\[
\delta_M(p) = \tau(p) - (\inf K(p)), \quad p \in M.
\]

For an integer $k \geq 0$, we denote by $S(n,k)$ the finite set of $k$–tuples $(n_1, \ldots, n_k)$ of integers $\geq 2$ satisfying $n_1 < n, n_1 + \cdots + n_k \leq n$. Denote by $S(n)$ the set of $k$–tuples with $k \geq 0$ for a fixed $n$.

For each $k$–tuple $(n_1, \ldots, n_k) \in S(n)$, Chen introduced a Riemannian invariant defined by
\[
\delta(n_1, \ldots, n_k)(p) = \tau(p) - \inf \{\tau(L_1) + \cdots + \tau(L_k)\},
\]
where $L_1, \ldots, L_k$ run over all $k$ mutually orthogonal subspaces of $T_pM$ such that $\dim L_j = n_j$, $j = 1, \ldots, k$.

The Chen inequalities provide fundamental relationships between the main extrinsic invariant (the squared mean curvature) and intrinsic invariants (Chen invariants).

We recall the most important Chen inequalities obtained by B.Y. Chen for submanifolds in real space forms.

**Theorem A [2].** Let $M$ be an $n$–dimensional ($n \geq 3$) submanifold of an $m$–dimensional real space form $\tilde{M}(c)$ of constant sectional curvature $c$. Then
\[
\delta_M \leq \frac{n - 2}{2} \left\{ \frac{n^2}{n - 1} \|H\|^2 + (n + 1)c \right\}. \tag{A}
\]

Equality holds if and only if, with respect to suitable frame fields $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$, the shape operators take the following forms:

\[
A_{n+1} = \begin{pmatrix}
  a & 0 & 0 & \cdots & 0 \\
  0 & \mu - a & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & \mu
\end{pmatrix},
\]

\[
A_r = \begin{pmatrix}
  h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\
  h_{12}^r & -h_{11}^r & 0 & \cdots & 0 \\
  0 & 0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad r = n + 2, \ldots, m.
\]

Furthermore, when the equality sign of (A) holds at a point $p \in M$, we also have $K(e_1 \wedge e_2) = \inf K$ at the point $p$. 
For each \((n_1, \ldots, n_k) \in S(n)\), we put:
\[
d(n_1, \ldots, n_k) = \frac{n^2 \left( n + k - 1 - \sum_{j=1}^{k} n_j \right)}{2 \left( n + k - \sum_{j=1}^{k} n_j \right)},
\]
\[
b(n_1, \ldots, n_k) = \frac{1}{2} \left[ n(n - 1) - \sum_{j=1}^{k} n_j(n_j - 1) \right].
\]

The following sharp inequality involving the Chen invariants and the squared mean curvature obtained in \([5]\) plays the most fundamental role in this topic.

**Theorem B.** For each \((n_1, \ldots, n_k) \in S(n)\) and each \(n\)-dimensional submanifold \(M\) in an \(m\)-dimensional Riemannian space form \(\tilde{M}(c)\) of constant sectional curvature \(c\), we have
\[
\delta(n_1, \ldots, n_k) \leq d(n_1, \ldots, n_k) \|H\|^2 + b(n_1, \ldots, n_k)c. \tag{B}
\]

The equality holds in (B) at a point \(p \in M\) if and only if there exists an orthonormal basis \(\{e_1, \ldots, e_m\}\) at \(p\) such that the shape operators of \(M\) in \(\tilde{M}(c)\) at \(p\) take the following forms:
\[
A_{n+1} = \begin{pmatrix}
    a_1 & 0 & 0 & \cdots & 0 \\
    0 & a_2 & 0 & \cdots & 0 \\
    0 & 0 & a_3 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & a_n
\end{pmatrix},
\]
\[
A_r = \begin{pmatrix}
    A'_{r-1} & \cdots & 0 & 0 & \cdots & 0 \\
    \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
    0 & \cdots & A'_{r-1} & 0 & \cdots & 0 \\
    0 & \cdots & 0 & \mu_r & \cdots & 0 \\
    \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
    0 & \cdots & 0 & 0 & \cdots & \mu_r
\end{pmatrix}, \quad r = n + 2, \ldots, m.
\]

where \(a_1, \ldots, a_n\) satisfy
\[
a_1 + \cdots + a_{n_1} = \cdots = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+\cdots+n_k} = a_{n_1+\cdots+n_k+1} = \cdots = a_n
\]
and each \(A'_{r}\) is a symmetric \(n_j \times n_j\) submatrix satisfying
\[
\text{trace}(A'_{r}) = \cdots = \text{trace}(A'_{r}) = \mu_r.
\]
2. Chen first inequality

The notion of a slant submanifold of an almost Hermitian manifold was introduced by B.Y. Chen [1].

**Definition.** A submanifold $M$ of an almost Hermitian manifold $\tilde{M}$ is said to be a slant submanifold if for any $p \in M$ and any nonzero vector $X \in T_pM$, the angle between $JX$ and the tangent space $T_pM$ is constant ($= \theta$).

In this paragraph, we prove the Chen first inequality for slant submanifolds $M$ in the generalized complex space forms $\tilde{M}(c, \alpha)$ ($RK$–manifolds of constant holomorphic sectional curvature $c$ and of constant type $\alpha$) [6].

We consider the Riemannian invariant $\delta'_M$ defined by

$$\delta'_M(p) = \tau(p) - \inf\{K(\pi)|\ \pi \text{ a plane section at } p \text{ invariant by } P\}.$$ 

**Theorem 2.1.** For any $2m$–dimensional generalized complex space form $\tilde{M}(c, \alpha)$ and a $\theta$–slant submanifold $M$, $\dim M = n, n \geq 3$, we have:

$$\delta'_M \leq \frac{n - 2}{2} \left\{ \frac{n^2}{n - 1} \|H\|^2 + (n + 1 + 3 \cos^2 \theta) \frac{c^4}{4} + (n + 1 - \cos^2 \theta) \frac{3\alpha^4}{4} \right\}. \quad (2.1)$$

The equality in (2.1) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of $T_pM$ and an orthonormal basis $\{e_{n+1}, \ldots, e_{2m}\}$ of $T_p^\perp M$ such that the shape operators of $M$ in $\tilde{M}(c, \alpha)$ at $p$ have the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, \quad a + b = \mu, \quad (2.2)$$

$$A_r = \begin{pmatrix} h_{11} & h_{12} & 0 & \cdots & 0 \\ h_{12} & -h_{11} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (2.3)$$
where we denote by
\[ A_r = A_{e_r}, \ r = n + 1, \ldots, 2m, \]  
(2.4)
\[ h_{ij}^r = g(h(e_i, e_j), e_r), \ i, j = 1, \ldots, n, \ r = n + 1, \ldots, 2m. \]  
(2.5)

**Proof.** We recall the Gauss equation for the submanifold \( M \).
\[
\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) \\
- g(h(X, Z), h(Y, W)), \ \forall X, Y, Z, W \in \Gamma TM,
\]  
(2.6)
where \( \tilde{R} \) denotes the curvature tensor of \( \tilde{M}(c, \alpha) \) and \( R \) denotes the curvature tensor of \( M \).

Since \( \tilde{M}(c, \alpha) \) is a generalized complex space form, its curvature tensor \( \tilde{R} \) is given by
\[
\tilde{R}(X, Y, Z, W) = \frac{c + 3\alpha}{4} \{ g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \} \\
+ \frac{c - \alpha}{4} \{ g(JX, Z)g(JY, W) - g(JX, W)g(JY, Z) \} \\
+ 2g(X, JY)g(Z, JW), \ \forall X, Y, Z, W \in \Gamma TM.
\]  
(2.7)

Let \( p \in M \), \( \{ e_1, e_2, \ldots, e_n \} \) be an orthonormal basis of \( T_pM \) and \( \{ e_{n+1}, \ldots, e_{2m} \} \) be an orthonormal basis of \( T^\perp_pM \). For \( X = Z = e_i, Y = W = e_j \), from the equation (2.7) it follows that
\[
\tilde{R}(e_i, e_j, e_i, e_j) = \frac{c + 3\alpha}{4} (n^2 - n) + \frac{c - \alpha}{4} \sum_{i,j} g^2(J_{e_i}e_j).
\]  
(2.8)

Let \( M \subset \tilde{M}(c, \alpha) \) be a \( \theta \)-slant submanifold, \( \dim M = n = 2k \). For \( X \in \Gamma TM \), we put \( JX = PX + FX \), with \( PX, FX \in \Gamma TM \).

For \( p \in M \), we choose an orthonormal basis \( \{ e_1, e_2, \ldots, e_n \} \) of \( T_pM \) with \( e_2 = (\sec \theta)Pe_1, \ldots, e_{2k} = (\sec \theta)Pe_{2k-1} \). It is easily seen that \( g(J_{e_i}e_{i+1}) = \cos \theta \), for \( i = 1, 3, \ldots, 2k - 1 \).

The relation (2.8) becomes
\[
\tilde{R}(e_i, e_j, e_i, e_j) = \frac{c + 3\alpha}{4} (n^2 - n) + \frac{c - \alpha}{4} 3n \cos^2 \theta.
\]  
(2.9)

Denoting, as usual,
\[
\|h\|^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)),
\]  
(2.10)
the relation (2.9) implies
\[
\frac{c + 3\alpha}{4} (n^2 - n) + \frac{e - \alpha}{4} n \cos^2 \theta = 2\tau + \|h\|^2 - n^2 \|H\|^2,
\]
(2.11)
or equivalently,
\[
2\tau = n^2 \|H\|^2 - \|h\|^2 + \frac{c + 3\alpha}{4} (n^2 - n) + \frac{e - \alpha}{4} n \cos^2 \theta.
\]
(2.12)
Putting
\[
\varepsilon = 2\tau - \frac{n^2}{n-1} (n-2) \|H\|^2 - \frac{c + 3\alpha}{4} (n^2 - n) - \frac{e - \alpha}{4} n \cos^2 \theta,
\]
(2.13)
we obtain
\[
n^2 \|H\|^2 = (n-1)(\varepsilon + \|h\|^2).
\]
(2.14)
Let \( p \in M, \pi \subset T_pM \) invariant by \( P \), \( \dim \pi = 2 \), \( \pi = \text{sp}\{e_1, e_2\} \). We choose \( e_{n+1} = \frac{H}{\|H\|} \) and from the relation (2.14) we obtain:
\[
\left( \sum_{i=1}^{n} h_{ii}^{n+1} \right)^2 = (n-1) \left( \sum_{i,j=1}^{n} \sum_{r=n+1}^{2m} (h_{ij}^r)^2 + \varepsilon \right),
\]
or equivalently,
\[
\left( \sum_{i=1}^{n} h_{ii}^{n+1} \right)^2 = (n-1) \left\{ \sum_{i=1}^{n} (h_{ii}^{n+1})^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n} \sum_{r=n+2}^{2m} (h_{ij}^r)^2 + \varepsilon \right\}.
\]
(2.15)
Now we recall the following algebraic lemma.

**Lemma** [2]. Let \( a_1, \ldots, a_n, c \) be \( n+1, n > 3 \), real numbers such that:
\[
\left( \sum_{i=1}^{n} a_i \right)^2 = (n-1) \left( \sum_{i=1}^{n} a_i^2 + c \right).
\]
Then \( 2a_1a_2 \geq c \). The equality holds if and only if \( a_1 + a_2 = a_3 = \cdots = a_n \).

By using the above Lemma, we have from (2.15):
\[
2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{i,j=1}^{n} \sum_{r=n+2}^{2m} (h_{ij}^r)^2 + \varepsilon.
\]
(2.16)
Gauss equation for $X = Z = e_1, Y = W = e_2$ gives

$$K(\pi) = \frac{c + 3\alpha}{4} + 3 \frac{c - \alpha}{4} \cos^2 \theta + \sum_{r=n+1}^{2m} \left[ h_{11}^r h_{22}^r - (h_{12}^r)^2 \right]$$

$$\geq \frac{c + 3\alpha}{4} + 3 \frac{c - \alpha}{4} \cos^2 \theta + \frac{1}{2} \sum_{i \neq j} (h_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + \varepsilon$$

$$+ \sum_{r=n+2}^{2m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m} (h_{12}^r)^2$$

$$= \frac{c + 3\alpha}{4} + 3 \frac{c - \alpha}{4} \cos^2 \theta + \frac{1}{2} \sum_{i \neq j} (h_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2$$

$$+ \frac{1}{2} \sum_{r=n+2}^{2m} (h_{11}^r + h_{22}^r)^2 + \sum_{j>2} \left[ (h_{ij}^{n+1})^2 + (h_{ij}^{n+1})^2 \right] + \frac{\varepsilon}{2}$$

$$\geq \frac{c + 3\alpha}{4} + 3 \frac{c - \alpha}{4} \cos^2 \theta + \frac{\varepsilon}{2},$$

which implies

$$K(\pi) \geq \frac{c + 3\alpha}{4} + 3 \frac{c - \alpha}{4} \cos^2 \theta + \frac{\varepsilon}{2}. \tag{2.17}$$

From the relation (2.12), it follows that:

$$\inf K \geq \frac{c + 3\alpha}{4} + 3 \frac{c - \alpha}{4} \cos^2 \theta + \tau - \frac{c + 3\alpha}{8} (n^2 - n)$$

$$- 3 \frac{c - \alpha}{8} n \cos^2 \theta - \frac{n^2(n-2)}{2(n-1)} \|H\|^2, \tag{2.18}$$

or equivalently

$$\delta'_M \leq \frac{n-2}{2} \left\{ \frac{n^2}{n-1} \|H\|^2 + (n+1+3 \cos^2 \theta) \frac{c}{4} + (n+1-\cos^2 \theta) \frac{3\alpha}{4} \right\}, \tag{2.19}$$

which is what we had to prove.

Equality at a point $p \in M$ holds if and only if equality holds in each of the previous inequalities and hence our Lemma yields equality.

$$\begin{cases}
    h_{ij}^{n+1} = 0, & \forall i \neq j, i, j > 2, \\
    h_{ij}^r = 0, & \forall i \neq j, i, j > 2, r = n+1, \ldots, 2m, \\
    h_{11}^r + h_{22}^r = 0, & \forall r = n+2, \ldots, 2m, \\
    h_{ij}^{n+1} = h_{ij}^{n+1} = 0, & \forall j > 2, \\
    h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \ldots = h_{nn}^{n+1}.
\end{cases}$$
We may choose \( \{e_1, e_2\} \) such that \( h^p_{12} = 0 \) and we denote \( a = h^1_{11} \), \( b = h^p_{22} \), \( \mu = h^p_{33} = \cdots = h^p_{nn} \).

It follows that the shape operators take the desired forms.

Using a similar method, we may prove the Chen inequality for totally real submanifolds in generalized complex space forms.

**Proposition 2.2.** For any \( 2m \)-dimensional generalized complex space form \( \tilde{M}(c, \alpha) \) and a totally real submanifold \( M \), \( \dim M = n, \ n \geq 3 \), we have

\[
\delta_M \leq \frac{n-2}{2} \left\{ \frac{n^2}{n-1} \|H\|^2 + \left( n + 1 + 3 \cos^2 \theta \right) \frac{c}{4} \right\}.
\]

The equality case of inequality (2.20) is identical with the equality case of inequality (2.1) from the Theorem 2.1.

**Remark.** For \( \alpha = 0 \) we obtain the Chen first inequality for slant submanifolds in complex space forms (see Theorem 3.2 from [9]).

**Corollary 2.3.** For any \( 2m \)-dimensional complex space form \( \tilde{M}(c) \) and a \( \theta \)-slant submanifold \( M \), \( \dim M = n, \ n \geq 3 \), we have:

\[
\delta' \leq \frac{n-2}{2} \left\{ \frac{n^2}{n-1} \|H\|^2 + \left( n + 1 + 3 \cos^2 \theta \right) \frac{c}{4} \right\}.
\]

3. General inequalities

Next we prove a generalization of the previous theorem, in terms of Chen invariants [6].

We consider the Riemannian invariant

\[
\delta'(n_1, \ldots, n_k)(p) = \tau(p) - \inf \{ \tau(L_1) + \cdots + \tau(L_k) \},
\]

where \( L_1, \ldots, L_k \) run over all \( k \) mutually orthogonal subspaces of \( T_p M \), invariant by \( P \), such that \( \dim L_j = n_j, \ j = 1, \ldots, k \).

**Theorem 3.1.** Let \( \tilde{M}(c, \alpha) \) be a generalized complex space form, \( \dim \tilde{M}(c) = 2m \) and \( M \) be an \( n \)-dimensional \( \theta \)-slant submanifold, \( n \geq 3 \), of \( \tilde{M}(c, \alpha) \). Let \( n_1, \ldots, n_k \) be integers \( \geq 2 \) satisfying \( n_1 < n, n_1 + \cdots + n_k \leq n \). Then, we have

\[
\delta'(n_1, \ldots, n_k) \leq d(n_1, \ldots, n_k) \|H\|^2 + b(n_1, \ldots, n_k) \frac{c + 3\alpha}{4} + \frac{3c}{8} \left( n - \sum_{j=1}^{k} n_j \right) \cos^2 \theta - \frac{3\alpha}{8} \left( n + \sum_{j=1}^{k} n_j \right) \cos^2 \theta.
\]

(3.1)
Let $\tilde{M}(c, \alpha)$ be a generalized complex space form, $\dim \tilde{M}(c, \alpha) = 2m$ and $M \subset \tilde{M}(c, \alpha)$ be an $n$–dimensional submanifold.

For any $p \in M$ and $X \in T_pM$, we have $JX = PX + FX, PX \in T_pM, FX \in T^\perp_pM$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_pM$. We put

$$\|P\|_2^2 = \sum_{i,j=1}^n g^2(Pe_i, e_j).$$

For an $r$–dimensional subspace $L$ of $T_pM$, we denote

$$\Psi(L) = \sum_{1 \leq i < j \leq r} g^2(Pu_i, u_j),$$

where $\{u_1, \ldots, u_r\}$ is an orthonormal basis of $L$.

In order to prove Theorem 3.1, we will use the following

**Lemma 3.2.** Let $\tilde{M}(c, \alpha)$ be a generalized complex space form, $\dim \tilde{M}(c, \alpha) = 2m$ and $M \subset \tilde{M}(c, \alpha)$ be an $n$–dimensional submanifold. Let $n_1, \ldots, n_k$ be integers $\geq 2$ satisfying $n_1 < n, n_1 + \cdots + n_k \leq n$. For $p \in M$, let $L_j \subset T_pM$ be subspaces of $T_pM$, invariant by $P$, such that $\dim L_j = n_j, \forall j = 1, \ldots, k$. Then we have:

$$\tau(p) - \sum_{j=1}^k \tau(L_j) \leq d(n_1, \ldots, n_k) \|H\|_2^2$$

$$+ \left[ n(n - 1) - \sum_{j=1}^k n_j(n_j - 1) \right] \frac{c}{8} - \left[ 3 \|P\|_2^2 - 6 \sum_{j=1}^k \Psi(L_j) \right] \frac{c}{8}$$

$$+ \left[ n(n - 1) - \sum_{j=1}^k n_j(n_j - 1) \right] \frac{3\alpha}{8} - \left[ \|P\|_2^2 + 2 \sum_{j=1}^k \Psi(L_j) \right] \frac{3\alpha}{8}. \quad (3.2)$$

**Proof.** Let $\tilde{M}(c, \alpha)$ be a $2m$–dimensional generalized complex space form and $M$ be an $n$–dimensional submanifold with $n \geq 3$.

Let $p \in M$ and $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_pM$; from Gauss equation, we obtain

$$2\tau = n^2 \|H\|_2^2 - \|k\|_2^2 + n(n - 1) \frac{c + 3\alpha}{4} + 3 \|P\|_2^2 \frac{c - \alpha}{4}. \quad (3.3)$$
Denoting by 

\[ \eta = 2\tau - 2d(n_1, \ldots, n_k) \| H \|^2 - n(n - 1) \frac{c + 3\alpha}{4} - 3 \| P \|^2 \frac{c - \alpha}{4}, \]

it follows that 

\[ n^2 \| H \|^2 = (\eta + \| h \|^2)\gamma, \]

where \( \gamma = n + k - \sum_{j=1}^{k} n_j. \)

Again the Gauss equation implies 

\[ \tau(L_j) = \left\{ n_j(n_j - 1) \frac{c + 3\alpha}{8} + 6\Psi(L_j) \right\} \frac{c - \alpha}{8} \]

\[ + \sum_{r=n+1}^{2m} \sum_{\alpha_j < \beta_j} [h_{\alpha_j\alpha_j}^{r} h_{\beta_j\beta_j}^{r} - (h_{\alpha_j\beta_j}^{r})^2]. \]

(3.6)

We will prove that 

\[ \sum_{j=1}^{k} \sum_{r=n+1}^{2m} \sum_{\alpha_j < \beta_j} [h_{\alpha_j\alpha_j}^{r} h_{\beta_j\beta_j}^{r} - (h_{\alpha_j\beta_j}^{r})^2] \geq \frac{\eta^2}{2}. \]

(3.7)

Let \( p \in M \) and \( \{e_1, \ldots, e_n\} \) be an orthonormal basis of \( T_pM \); let \( L_1, \ldots, L_k \) be \( k \) mutually orthogonal subspaces of \( T_pM \), \( \dim L_j = n_j \), defined by 

\[ L_1 = \text{sp} \{e_1, \ldots, e_{n_1}\}, \]

\[ L_2 = \text{sp} \{e_{n_1+1}, \ldots, e_{n_1+n_2}\}, \]

\[ \vdots \]

\[ L_k = \text{sp} \{e_{n_1+\cdots+n_{k-1}+1}, \ldots, e_{n_1+\cdots+n_{k-1}+n_k}\}. \]

Let \( e_{n+1} = \frac{h_{n_1+1}}{\sqrt{\eta}}, e_{n+1} \in T_p^\perp M. \) We put \( a_i = h_{ii}^{n+1} = g(h(e_i, e_i), e_{n+1}). \) The relation (3.5) becomes 

\[ \left( \sum_{i=1}^{n} a_i \right)^2 = \gamma \left[ \eta + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{i=1}^{n} (h_{ii}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i=1}^{n} (h_{ij}^{r})^2 \right]. \]

(3.8)

We denote by \( D_j, j = 1, \ldots, k, \) the sets: 

\[ D_1 = \{1, \ldots, n_1\}, \]

\[ D_2 = \{n_1 + 1, \ldots, n_1 + n_2\}, \]

\[ \vdots \]

\[ D_k = \{n_1 + \cdots + n_{k-1} + 1, \ldots, n_1 + \cdots + n_{k-1} + n_k\}. \]
Also, we put
\[ b_1 = a_1, \]
\[ b_2 = a_2 + \cdots + a_{n_1}, \]
\[ b_3 = a_{n_1+1} + \cdots + a_{n_1+n_2}, \]
\[ \vdots \]
\[ b_{k+1} = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+n_2+\cdots+n_{k-1}+n_k}, \]
\[ b_{k+2} = a_{n_1+\cdots+n_k+1}, \]
\[ \vdots \]
\[ b_{\gamma+1} = a_n. \]

Then the relation (3.8) can be written as
\[ \left( \sum_{i=1}^{\gamma+1} b_i \right)^2 = \gamma \left[ \eta + \sum_{i=1}^{\gamma+1} b_i^2 + \sum_{i \neq j} (h_i^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{j=1}^n (h_r^j)^2 \right. \]
\[ \left. - 2 \sum_{\alpha < \beta_1} a_{\alpha_1} a_{\beta_1} - \cdots - 2 \sum_{\alpha_k < \beta_k} a_{\alpha_1} a_{\beta_k} \right], \]

with \( \alpha_j, \beta_j \in D_j, \forall j = 1, \ldots, k. \)

Applying the algebraic Lemma from section 2, we have
\[ \sum_{\alpha < \beta_1} a_{\alpha_1} a_{\beta_1} + \cdots + \sum_{\alpha_k < \beta_k} a_{\alpha_k} a_{\beta_k} \geq \frac{1}{2} \left[ \eta + \sum_{i \neq j} (h_i^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{j=1}^n (h_r^j)^2 \right]. \]

It follows that
\[ \sum_{j=1}^k \sum_{r=n+1}^{2m} \sum_{\alpha_j < \beta_j} \left( h_r^{\alpha_j} h_r^{\beta_j} - (h_r^r)^2 \right) \]
\[ \geq \frac{\eta}{2} + \frac{1}{2} \sum_{r=n+1}^{2m} \sum_{(\alpha, \beta) \notin D^2} (h_r^r)^2 + \frac{1}{2} \sum_{r=n+2}^{2m} \sum_{j=1}^k \left( \sum_{\alpha_j \in D_j} h_r^{\alpha_j} \right)^2 \geq \frac{\eta}{2}, \]

where we denote by \( D^2 = (D_1 \times D_1) \cup \cdots \cup (D_k \times D_k). \)

Thus we have proved the relation (3.7). From the relation (3.6) we obtain
\[ \sum_{j=1}^k \tau(L_j) \geq \frac{\eta}{2} + \sum_{j=1}^k n_j (n_j - 1) \frac{c + 3\alpha}{8} + \sum_{j=1}^k \Psi(L_j) \frac{c - \alpha}{8} \]
\[ = \tau - d(n_1, \ldots, n_k) \| H \| - n(n-1) \frac{c + 3\alpha}{8} - 3 \| P \| \frac{c - \alpha}{8} \]
\[ + \sum_{j=1}^k n_j (n_j - 1) \frac{c + 3\alpha}{8} + \sum_{j=1}^k \Psi(L_j) \frac{c - \alpha}{8} \],
or equivalently,
\[
\tau - \sum_{j=1}^{k} \tau(L_j) \leq d(n_1, \ldots, n_k) \|H\|^2 + \frac{c}{4} \left\{ \frac{n(n - 1) - \sum_{j=1}^{k} n_j(n_j - 1)}{2} \right\} + 3 \alpha \left[ \frac{\|P\|^2}{8} - \frac{3}{4} \sum_{j=1}^{k} \Psi(L_j) \right]
\]
which is what we had to prove.

**Proof of Theorem 3.1.** Let $\tilde{M}(c, \alpha)$ be a generalized complex space form, $\dim \tilde{M}(c, \alpha) = 2m$ and $M \subset \tilde{M}(c, \alpha)$ be an $n$-dimensional $\theta$–slant submanifold, $n \geq 3$, $n = 2k$. Let $p \in M$ and $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_pM$.

Since we use subspaces invariant by $P$, we may choose $e_2 = (\sec \theta) Pe_1, \ldots, e_{2k} = (\sec \theta) Pe_{2k-1}$. It is easily seen that $\|P\|^2 = n \cos^2 \theta$.

Let $p \in M$ and $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_pM$; let $L_1, \ldots, L_k$ be $k$ mutually orthogonal subspaces of $T_pM$, $\dim L_j = n_j$, defined by
\[
L_1 = \text{sp} \{e_1, \ldots, e_{n_1}\}, \\
L_2 = \text{sp} \{e_{n_1+1}, \ldots, e_{n_1+n_2}\}, \\
\vdots \\
L_k = \text{sp} \{e_{n_1+\cdots+n_{k-1}+1}, \ldots, e_{n_1+\cdots+n_{k-1}+n_k}\}.
\]

In the same way, it follows that $\Psi(L_j) = \frac{n_j}{2} \cos^2 \theta$, $\forall j = 1, \ldots, k$.

From (3.2) we obtain the inequality (3.1).

Similarly, we may prove

**Proposition 3.3.** For any $2m$–dimensional generalized complex space form $\tilde{M}(c, \alpha)$ and a totally real submanifold $M$, $\dim M = n$, $n \geq 3$, we have
\[
\delta(n_1, \ldots, n_k) \leq d(n_1, \ldots, n_k) \|H\|^2 + b(n_1, \ldots, n_k) \frac{c + 3\alpha}{4}.
\]

The following corollary represents the Chen inequalities for slant submanifolds in complex space forms (see [9]).
**Corollary 3.4.** For any $2m$–dimensional complex space form $\tilde{M}(c)$ and a $\theta$–slant submanifold $M$, $\dim M = n, n \geq 3$, we have

$$\delta'(n_1, \ldots, n_k) \leq d(n_1, \ldots, n_k) \|H\|^2 + b(n_1, \ldots, n_k) \frac{c}{4} + \frac{3}{8} \left( n - \sum_{j=1}^{k} n_j \right) \cos^2 \theta.$$  

**REFERENCES**


Nejednakosti B.Y. Chena za “slant” podmnogostrukosti u generaliziranim oblicima kompleksnog prostora

Adela Mihai

Sadržaj

U radu je uspostavljena nejednakost Chena za “slant” podmnogostrukosti $M$ u generaliziranim oblicima kompleksnog prostora $\tilde{M}(c, \alpha)$ ($RK$-mnogostrukosti konstantne holomorfne sekcionalne zakrivljenosti $c$ i konstantnog tipa $\alpha$).