ON THE DUAL BASIS OF PROJECTIVE SEMIMODULES
AND ITS APPLICATIONS

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Abstract. The dual basis lemma for projective semimodule over a
semiring is proved. We show under which conditions the two categories
cs mod $\text{-} R$ and $S \text{-} cs mod$ of cancellative semimodules are equivalent
and how these equivalences are realized.

1. Introduction

Projective semimodules over semirings are characterized in [2]. Here we
generalize one of the classical tools from the theory of modules over rings
called the dual basis lemma, for projective semimodule over a semiring. We
define generator and progenerator semimodules over semirings and show
under which conditions the two categories $cs mod - R$ and $S - cs mod$ of
cancellative semimodules are equivalent and how such equivalences are real-
ized.

2. Results

Dual Basis Lemma. Let $M$ be an $R$–semimodule. Then $M$ is projective
if and only if there exists $\{m_i\}_{i \in I} \subset M$ and $\{f_i\}_{i \in I} \subset \text{Hom}_R(M, R)$ ($I$ some
index set) such that

a) for every $m \in M$, $f_i(m) = 0$ for all but finitely many $i \in I$ and
b) for every $m \in M$, $\sum_{i \in I} f_i(m) m_i = m$.

The collection $\{m_i, f_i\}$ is called a dual basis for $M$.

Proof. Let $R^{(I)}$ be a free $R$–semimodule and $\theta$ be a surjective $R$–ho-
momorphism from $R^{(I)}$ to $M$ where $R^{(I)}$ is the set of all functions from $I$ to $R$
with finite support.

Since $M$ is a projective semimodule, there exists an $R$– homomorphism
$\psi : M \to R^{(I)}$ such that $\theta \psi = \text{Id}_M$. Let $\pi_i : R^{(I)} \to R$ be given by

2000 Mathematics Subject Classification. 16Y 60.
\[ \pi_i(f) = f(i) \] for all \( f \in R(I) \), then for any \( f \) in \( R(I) \) we have \( \sum_{i \in I} \pi_i(f)e_i = f \), since \( \left[ \sum_{i \in I} \pi_i(f)e_i \right](j) = \pi_j(f) = f(j) \) where \( e_i \in R(I) \) defined by

\[
e_i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}
\]

Now set \( m_i = \theta(e_i) \) and \( \pi_i \psi = f_i \). For \( m \in M \) clearly \( f_i(m) = 0 \), for all but finitely many \( i \).

Now,

\[
\sum_{i \in I} f_i(m)m_i = \sum_{i \in I} (\pi_i \psi)(m)m_i
= \sum_{i \in I} \pi_i(\psi(m))\theta(e_i)
= \theta \left( \sum_{i \in I} \pi_i(\psi(m)) \right)(e_i)
= \theta(\psi(m))
= (\theta \psi)(m)
= m, \text{ for all } m \in M.
\]

Thus \( \{ m_i, f_i \} \) forms a dual basis for \( M \).

Conversely, suppose that \( \{ m_i, f_i \} \) is a dual basis for \( R\text{-semimodule } M \).

Define \( \psi : M \to R(I) \) by \( \psi(m)(i) = f_i(m) \) for all \( m \in M \) and \( \theta : R(I) \to M \) by \( \theta(f) = \sum_{i \in I} f(i)m_i \) for \( m \in M \) and \( f \in R(I) \). Then \( \theta \) and \( \psi \) are \( R\text{-homomorphisms of left } R\text{-semimodules and}

\[
(\theta \psi)(m) = \theta(\psi(m))
= \theta(f_i(m))
= \sum_{i \in I} f_i(m)m_i
= m, \text{ for all } m \in M.
\]

Let \( \phi : L \to K \) be a surjective \( R\text{-homomorphism of left } R\text{-semimodules and } \alpha : M \to K \) be an \( R\text{-homomorphism. Since } R(I) \text{ is projective, then there exists an } R\text{-homomorphism } \beta : R(I) \to L \text{ such that } \phi \beta = \alpha \theta \Rightarrow \phi \beta \psi = \alpha \theta \psi = \alpha \text{ and } \beta \psi : M \to L \text{ is a map having the property that we seek in order to prove the first condition of projectiveness. Now let } \phi : L \to K \text{ be a steady } R\text{-homomorphism of left } R\text{-semimodules and } \alpha, \alpha' : M \to L \text{ be } R\text{-homomorphisms satisfying } \phi \alpha = \phi \alpha' \text{ which implies that } \phi \alpha \theta = \phi \alpha' \theta. \)
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Since $R(I)$ is projective, there exist $R$–homomorphisms $\beta, \beta' : R(I) \to L$ satisfying $\phi \beta = \phi \beta'$ and $\alpha \theta + \beta = \alpha' \theta + \beta'$. This implies $\phi(\beta \psi) = \phi(\beta' \psi)$ and $\alpha + \beta \psi = \alpha \theta \psi + \beta \psi = (\alpha \theta + \beta) \psi = (\alpha' \theta + \beta') \psi = \alpha' + \beta' \psi$.

Hence the second condition of projectiveness. \hfill \Box

Tensor product is as defined in [2]. Note that if $M$ is a cancellative left $R$–semimodule then $R \otimes M \cong M$.

**Proposition 1.** Let $R$ be a cancellative semiring and $M$ be a cancellative $R$–semimodule. Then $\text{Hom}_R(R, M) \cong M$.

**Proposition 2.** Let $R$ be a commutative semiring and let $A$ and $B$ be $R$–semialgebras. Let $M$ be a finitely generated and projective $A$–semimodule and let $N$ be a finitely generated and projective $B$–semimodule. Then

$$\text{Hom}_A(M, M) \otimes \text{Hom}_B(N, N) \cong \text{Hom}_{A \otimes B}(M \otimes N, M \otimes N)$$

where $\otimes = \otimes_R$.

**Proof.** Let $\{x_j, f_j\}, \{y_i, g_i\}$ be the dual bases for $M$ and $N$ respectively. Then for any $m$ in $M$ and $n$ in $N$, $\sum_j f_j(m) x_j = m$ and $\sum_i g_i(n) y_i = n$.

Define,

$$\theta_j : M \otimes \text{Hom}_B(N, N) \to \text{Hom}_A(M, M) \otimes \text{Hom}_B(N, N)$$

by

$$\theta_j(a \otimes h) = f_j(\ ) a \otimes h$$

and

$$\pi_i : M \otimes N \to M \otimes \text{Hom}_B(N, N)$$

given by

$$\pi_i(b_1 \otimes b_2) = b_1 \otimes g_i(\ ) b_2.$$

Now define

$$\psi : \text{Hom}_{A \otimes B}(M \otimes N, M \otimes N) \to \text{Hom}_A(M, M) \otimes \text{Hom}_B(N, N)$$

by

$$\psi(f) = \sum_{i,j} \theta_j(\pi_i[f(x_j \otimes y_i)])$$

and

$$\psi' : \text{Hom}_A(M, M) \otimes \text{Hom}_B(N, N) \to \text{Hom}_{A \otimes B}(M \otimes N, M \otimes N)$$

by

$$\psi'(h_1 \otimes h_2) = h_1 \otimes h_2.$$
Consider,
\[
\psi \psi'(h_1 \otimes h_2)(m \otimes n) = \psi(h_1 \otimes h_2)(m \otimes n)
\]
\[
= \sum_{i,j} \theta_j(\pi_i[h_1 \otimes h_2(x_j \otimes y_i)])(m \otimes n)
\]
\[
= \sum_{i,j} \theta_j(\pi_i[(h_1(x_j) \otimes h_2(y_i)])(m \otimes n))
\]
\[
= \sum_{i,j} \theta_j(h_1(x_j) \otimes g_i( h_2(y_i))(m \otimes n)
\]
\[
= (h_1 \otimes h_2)\left(\sum_{i,j} f_j(m)x_j \otimes g_i(n)y_i\right)
\]
\[
= (h_1 \otimes h_2)(m \otimes n)
\]
\[
\Rightarrow \psi \psi'(h_1 \otimes h_2)(m \otimes n) = (h_1 \otimes h_2)(m \otimes n).
\]

Clearly, \(\psi \psi(f) = f\). Hence \(\psi\) is a one–one onto homomorphism.

For any \(R\)–semimodule \(M\), consider the subset \(I_R(M)\) of \(R\) consisting of the element of the form \(\sum_{i=1}^n f_i(m_i)\) where the \(f_i \in \text{Hom}_R(M, R)\) and the \(m_i \in M\). The \(I_R(M)\) is two–sided ideal in \(R\) so \(I_R(M)\) is an ideal in \(R\) and is called the **trace ideal** of \(M\). An \(R\)–semimodule \(M\) is an **\(R\)–generator** if \(I_R(M) = R\). Thus \(M\) is an \(R\)–generator if and only if there exist \(f_1, f_2, \ldots, f_n \in \text{Hom}_R(M, R)\) and \(m_1, m_2, \ldots, m_n \in M\) with \(\sum_{i=1}^n f_i(m_i) = 1\).

An \(R\)–semimodule \(M\) is an **\(R\)–progenerator** if \(M\) is a finitely generated, projective and generator over \(R\).

**Proposition 3.** Let \(R\) be a commutative semiring and let \(M\) and \(N\) be \(R\)–semimodules. Then

i) \(M \otimes_R N\) is finitely generated over \(R\) if both \(M\) and \(N\) are.

ii) \(M \otimes_R N\) is \(R\)–projective if both \(M\) and \(N\) are.

iii) \(M \otimes_R N\) is \(R\)–generator if both \(M\) and \(N\) are.

Henceforth we show that **\(csmod\)–\(R\)** and **\(S\)–\(csmod\)** are equivalent categories where \(S\) is chosen as the cancellative semiring of endomorphisms of some cancellative **\(R\)–progenerator**.

Let \(R\) be any cancellative semiring and let \(M\) be any cancellative \(R\)–semimodule. Define \(M^* = \text{Hom}_R(M, R)\) and \(S = \text{Hom}_R(M, M)\). Note that \(M^*, S\) are cancellative. Since \(R\) is a cancellative \((R \otimes R)\) bisemimodule, \(M^*\) is a cancellative right \(R\)–semimodule under the operation \((f \cdot r)m = f(m)r\).

Moreover \(M\) is a cancellative left \(S\)–semimodule with \(s \cdot m = s(m)\). Under this operation \(M\) is a cancellative left \(R\)–left \(S\) bisemimodule. Hence \(M^*\) becomes a cancellative right \(S\)–semimodule under the operation \((f \cdot s)(m) =
$f(s(m))$. $M^*$ is a cancellative right $R$–right $S$–bisemimodule. We can form $M^* \otimes_R M$ and $M^* \otimes_S M$. Moreover $M^* \otimes_R M$ is a cancellative left $S$–right $S$–bisemimodule by virtue of $M$ being a cancellative left $R$–left $R$–bisemimodule and $M^*$ being a cancellative right $R$–right $S$–bisemimodule. Similarly $M^* \otimes_S M$ is a cancellative left $R$–right $R$ bisemimodule.

Let $\theta_R$ denote the map from $M^* \otimes_R M$ to $S = \text{Hom}_R(M, M)$ given by $[\theta_R \sum_i (f_i \otimes m_i)](m) = \sum_i f_i(m)m_i$. $\theta_R$ is both a left and a right $S$–semimodule homomorphism. Let $\theta_S$ denote the map from $M^* \otimes_S M$ to $R$ given by $\theta_S(\sum_i f_i \otimes m_i) = \sum_i f_i(m_i)$. $\theta_S$ is a right and left $R$–semimodule homomorphism, whose image is the trace ideal $I_R(M)$.

**Lemma 1.** Let $R$ be any cancellative semiring and $M$ be any cancellative $R$–semimodule. $\theta_R$ is onto iff $M$ is finitely generated and projective. Moreover if $\theta_R$ is onto then it is one–one.

**Proof.** Suppose that $M$ is finitely generated and projective. Therefore there exists a dual basis $f_1, f_2, \ldots, f_n \in M^*$ and $m_1, m_2, \ldots, m_n \in M$, such that $\theta_R[\sum_{i=1}^{n} (f_i \otimes m_i)](m) = g$ for any $g$ in $S = \text{Hom}_R(M, M)$. Hence $\theta_R$ is onto.

Conversely, assume that $\theta_R$ is onto. Then there exist $\sum_{i=1}^{n} f_i \otimes m_i \in M^* \otimes_R M$ such that $\theta_R(\sum_{i=1}^{n} f_i \otimes m_i)$ is the identity map from $M$ to $M$, that is, $\sum_{i=1}^{n} f_i(m)m_i = m$ for all $m \in M$.

Thus the set $f_1, f_2, \ldots, f_n$, and $m_1, m_2, \ldots, m_n$ forms a finite dual basis for $M$. Therefore by the dual basis lemma, $M$ is finitely generated and projective.

Now given that $\theta_R$ is onto, we know that $M$ possesses a dual basis $f_1, f_2, \ldots, f_n \in M^*$ and $m_1, m_2, \ldots, m_n \in M$.

We claim that $\theta_R$ is one–one. Let $\sum_j g_j \otimes n_j, \sum_k h_k \otimes p_k \in M^* \otimes_R M$ satisfy

$$\theta_R\left(\sum_j g_j \otimes n_j\right)(m) = \theta_R\left(\sum_k h_k \otimes p_k\right)(m), \ \forall m \in M.$$  

Then

$$\sum_j g_j(m)n_j = \sum_k h_k(m)p_k.$$  

Now

$$\sum_j g_j \otimes n_j = \sum_j g_j \otimes \left(\sum_i f_i(n_j)\right)m_i = \sum_{i,j} g_j f_i(n_j) \otimes m_i.$$  

But

$$\sum_j (g_j f_i(n_j))(m) = \sum_j (g_j(f_i(n_j))(m))$$
\[
\sum_j (g_j(m)f_i(n_j)) = f_i\left(\sum_j g_j(m)\right) = f_i\left(\sum_k h_k(m)\right) = \sum_k h_k(m)f_i(p_k) = \sum_k (h_kf_i(p_k))(m).
\]

Therefore

\[
\left[\sum_j g_jf_i(n_j)\right](m) = \left[\sum_k h_kf_i(p_k)\right](m), \quad \forall m \in M
\]

\[
\Rightarrow \sum_j g_jf_i(n_j) = \sum_k h_kf_i(p_k)
\]

\[
\Rightarrow \sum_{i,j} g_jf_i(n_j) \otimes m_i = \sum_{i,k} h_kf_i(p_k) \otimes m_i
\]

\[
\Rightarrow \sum_j g_j \otimes n_j = \sum_k h_k \otimes p_k.
\]

Thus

\[
\theta_R\left(\sum_j g_j \otimes n_j\right) = \theta_R\left(\sum_k h_k \otimes p_k\right)
\]

\[
\Rightarrow \sum_j g_j \otimes n_j = \sum_k h_k \otimes p_k.
\]

Hence \(\theta_R\) is one-one. \(\square\)

**Lemma 2.** Let \(R\) be any cancellative semiring, \(M\) be any cancellative \(R\)-semimodule and \(S = \text{Hom}_R(M, M)\) be a cancellative semiring. \(\theta_S\) is onto if and only if \(M\) is a generator. Moreover if \(\theta_S\) is onto then it is one-one.

**Proof.** Since the image of \(\theta_S\) is equal to \(I_R(M)\), \(\theta_S\) is onto if and only if \(I_R(M) = R\), that is \(M\) is a generator over \(R\).

Suppose \(\theta_S\) is onto. We claim that \(\theta_S\) is one-one. Let \(\sum_j h_j \otimes n_j, \sum_k g_k \otimes p_k \in M^* \otimes_S M\) satisfy

\[
\theta_S\left(\sum_j h_j \otimes n_j\right) = \theta_S\left(\sum_k g_k \otimes p_k\right).
\]
Then

\[ \sum_j h_j(n_j) = \sum_k g_k(p_k). \]

Since \( \theta_S \) is onto, there exist \( f_1, f_2, \ldots, f_n \in M^* \) and \( m_1, m_2, \ldots, m_n \in M \) with

\[ \sum_i f_i(m_i) = 1. \]

Now

\[ \sum_j h_j \otimes n_j = \sum_j h_j \otimes \left( \sum_i f_i(m_i) \right) n_j \]
\[ = \sum_{i,j} h_j \otimes \theta_R(f_i \otimes n_j)(m_i) \]
\[ = \sum_i \left( \sum_j h_j \theta_R(f_i \otimes n_j) \right) \otimes m_i. \]

Note that for every \( i \) and every \( m \in M \),

\[ \left[ \sum_j h_j \theta_R(f_i \otimes n_j) \right](m) = \sum_j h_j(f_i(m)n_j) \]
\[ = f_i(m) \left( \sum_j h_j(n_j) \right) \]
\[ = f_i(m) \left( \sum_k g_k(p_k) \right) \]
\[ = \sum_k g_k(f_i(m)p_k) \]
\[ = \left[ \sum_k g_k \theta_R(f_i \otimes p_k) \right](m). \]

Therefore

\[ \left[ \sum_j h_j \theta_R(f_i \otimes n_j) \right](m) = \left[ \sum_k g_k \theta_R(f_i \otimes p_k) \right](m), \forall m \in M. \]

So,

\[ \left[ \sum_j h_j \theta_R(f_i \otimes n_j) \right] = \left[ \sum_k g_k \theta_R(f_i \otimes p_k) \right] \]
\[ \Rightarrow \left[ \sum_{i,j} h_j \theta_R(f_i \otimes n_j) \right] \otimes m_i = \left[ \sum_{i,k} g_k \theta_R(f_i \otimes p_k) \right] \otimes m_i \]
\[ \Rightarrow \sum_j h_j \otimes n_j = \sum_k g_k \otimes p_k. \]
Thus

\[ \theta_S \left( \sum_j h_j \otimes n_j \right) = \theta_S \left( \sum_k g_k \otimes p_k \right) \]

\[ \Rightarrow \sum_j h_j \otimes n_j = \sum_k g_k \otimes p_k. \]

Hence \( \theta_S \) is one–one.

For any cancellative left \( R \)–semimodule \( M \), we have seen that \( M \) is a left \( R \)–left \( S \) cancellative bisemimodule and \( M^* = \text{Hom}_R(M, R) \) is a right \( R \)–right \( S \) cancellative bisemimodule where \( S = \text{Hom}_R(M, M) \) is a cancellative semiring. Therefore for any cancellative right \( R \)–semimodule \( L \), \( L \otimes_R M \) has the structure of a left cancellative \( S \)–semimodule, while for any cancellative left \( S \)–semimodule \( N \), \( M^* \otimes_S N \) has the structure of a cancellative right \( R \)–semimodule.

Then

\[ ( \ ) \otimes_R M : \text{cs mod } -R \rightarrow S - \text{cs mod} \]

and

\[ M^* \otimes_S ( \ ) : S - \text{cs mod} \rightarrow \text{cs mod } -R \]

are functors.

**Theorem 4.** Let \( R \) be any cancellative semiring, \( M \) be any cancellative left \( R \)–semimodule and left \( R \) progenerator. Consider the cancellative semiring \( S = \text{Hom}_R(M, M) \) and the cancellative semimodule \( M^* = \text{Hom}_R(M, R) \). Then the functors

\[ ( \ ) \otimes_R M : \text{cs mod } -R \rightarrow S - \text{cs mod}, \]

\[ M^* \otimes_S ( \ ) : S - \text{cs mod} \rightarrow \text{cs mod } -R \]

are inverse equivalences.

**Proof.** Let \( L \in \text{cs mod } -R \). Then we have

\[ M^* \otimes_S (L \otimes_R M) \cong M^* \otimes_S (M \otimes_R L) \]

\[ \cong (M^* \otimes_S M) \otimes_R L \]

\[ \cong (R \otimes_R L). \]

\[ \cong L \otimes_R R \cong L. \]

Similarly for any cancellative left \( S \)–semimodule \( N \),

\[ (M^* \otimes_S N) \otimes_R M \cong (N \otimes_S M^*) \otimes_R M \]

\[ \cong N \otimes_S (M^* \otimes_R M) \]

\[ \cong N \otimes_S S \]

\[ \cong S \otimes_S N \cong N. \]
Hence the functors are inverse equivalences.

Acknowledgement. The authors would like to thank the referee and editor in chief for useful suggestions for the improvement of the article.

References


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