GENERAL SUMMABILITY FACTOR THEOREMS AND APPLICATIONS

B. E. RHoadES AND EKREM SAVAŞ

Abstract. We obtain sufficient and (different) necessary conditions for the series \( \sum a_n \), which is absolutely summable of order \( k \) by a triangular matrix method \( A \), to be such that \( \sum a_n \lambda_n \) is absolutely summable of order \( k \) by a triangular matrix \( B \). As corollaries we obtain a number of inclusion theorems.

In a recent paper the authors [3] obtained sufficient conditions for a series \( \sum a_n \) which is absolutely summable of order \( k \) by a weighted mean method to be such that \( \sum a_n \lambda_n \) is absolutely summable of order \( k \) by a triangular matrix method. In this paper we establish a more general summability factor theorem involving two lower triangular matrices. Using these results we obtain a number of corollaries.

Let \( T \) be a lower triangular matrix, \( \{s_n\} \) a sequence. Then

\[
T_n := \sum_{\nu=0}^{n} t_{n\nu} s_\nu.
\]

A series \( \sum a_n \) is said to be summable \( |T|_k, k \geq 1 \) if

\[
\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty. \quad (1)
\]

We may associate with \( T \) two lower triangular matrices \( \bar{T} \) and \( \hat{T} \) as follows:

\[
\bar{t}_{n\nu} := \sum_{r=\nu}^{n} t_{nr}, \quad n, \nu = 0, 1, 2, \ldots,
\]

and

\[
\hat{t}_{n\nu} := \bar{t}_{n\nu} - \bar{t}_{n-1,\nu}, \quad n = 1, 2, 3, \ldots.
\]

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With \( s_n := \sum_{i=0}^{n} \lambda_i a_i \).

\[
y_n := \sum_{i=0}^{n} t_{ni} s_i = \sum_{i=0}^{n} t_{ni} \sum_{\nu=0}^{i} \lambda_\nu a_\nu = \sum_{\nu=0}^{n} \lambda_\nu a_\nu \sum_{i=\nu}^{n} t_{ni} = \sum_{\nu=0}^{n} \hat{t}_{n\nu} \lambda_\nu a_\nu
\]

and

\[
Y_n := y_n - y_{n-1} = \sum_{\nu=0}^{n} (\hat{t}_{n\nu} - \hat{t}_{n-1,\nu}) \lambda_\nu a_\nu = \sum_{\nu=0}^{n} \hat{t}_{n\nu} \lambda_\nu a_\nu.
\]  

We shall call \( T \) a triangle if \( T \) is lower triangular and \( t_{nn} \neq 0 \) for each \( n \).

The notation \( \Delta_\nu \hat{a}_{n\nu} \) means \( \hat{a}_{n\nu} - \hat{a}_{n,\nu+1} \).

Theorem 1 of this paper represents the first time that two arbitrary triangles have been used in a summability factor theorem for absolute summability of order \( k > 1 \). By restricting \( A \) and \( B \) to be specific matrices we obtain summability factor theorems for specific classes of matrices, such as weighted means and the Cesàro matrix of order 1. By setting each \( \lambda_n = 1 \) we obtain a number of inclusion theorems.

The notation \( \lambda \in (|A|_k, |B|_k) \) will be used to represent the statement that, if \( \sum a_n \) is summable \( |A|_k \), then \( \sum a_n \lambda_n \) is summable \( |B|_k \).

**Theorem 1.** Let \( \{\lambda_n\} \) be a sequence of constants, \( A \) and \( B \) triangles satisfying

1. \( \frac{|b_{nn}|}{|a_{nn}|} = O\left(\frac{1}{|\lambda_n|}\right) \),
2. \( |a_{nn} - a_{n+1,n}| = O(|a_{nn}a_{n+1,n+1}|) \),
3. \( \sum_{\nu=0}^{n-1} |\Delta_\nu (\hat{b}_{n\nu} \lambda_\nu)| = O(|b_{nn} \lambda_n|) \),
4. \( \sum_{n=\nu+1}^{\infty} |n| |b_{nn} \lambda_n|^{k-1} |\Delta_\nu (\hat{b}_{n\nu} \lambda_\nu)| = O(\nu^{k-1} |b_{\nu\nu} \lambda_\nu|^k) \),
5. \( \sum_{\nu=0}^{n-1} |b_{\nu\nu}| |\hat{b}_{n,\nu+1} \lambda_{\nu+1}| = O(|b_{nn} \lambda_{n+1}|) \),
6. \( \sum_{n=\nu+1}^{\infty} |n| |b_{nn} \lambda_{n+1}|^{k-1} |\hat{b}_{n,\nu+1}| = O((\nu |b_{\nu\nu} \lambda_{\nu+1}|)^{k-1}) \),
7. \( \sum_{\nu=1}^{\infty} \nu^{k-1} |\lambda_{\nu+1} X_\nu|^k = O(1) \),
\( (\text{viii}) \sum_{n=1}^{\infty} n^{k-1} \left| \sum_{\nu=2}^{n} \hat{b}_{\nu \nu} \lambda_{\nu} \sum_{i=0}^{\nu-2} \hat{a}_{\nu i} X_i \right|^2 = O(1) \)

where \( X_{\nu}, X_i \) and \( \hat{a}_{\nu i} \) are defined latter, in formulas (4) and (5).

Then \( \lambda \in (|A|_k, |B|_k) \).

**Proof.** If \( y_n \) denotes the \( n \)-th term of the \( B \)-transform of a sequence \( \{s_n\} \), then

\[
y_n = \sum_{i=0}^{n} b_{ni} s_i = \sum_{i=0}^{n} b_{ni} \sum_{\nu=0}^{i} \lambda_{\nu} a_{\nu} = \sum_{\nu=0}^{n} \lambda_{\nu} a_{\nu} \sum_{i=0}^{\nu} b_{ni} = \sum_{\nu=0}^{n} \hat{b}_{\nu \nu} \lambda_{\nu} a_{\nu}.
\]

\[
y_{n-1} = \sum_{\nu=0}^{n-1} \hat{b}_{n-1,\nu} \lambda_{\nu} a_{\nu}.
\]

\[
Y_n := y_n - y_{n-1} = \sum_{\nu=0}^{n} \hat{b}_{\nu \nu} \lambda_{\nu} a_{\nu}, \quad (3)
\]

where \( s_n = \sum_{i=0}^{n} \lambda_{i} a_{i} \).

Let \( x_n \) denote the \( n \)-th term of the \( A \)-transform of a series \( \sum a_n \). Then

\[
X_n := x_n - x_{n-1} = \sum_{\nu=0}^{n} \hat{a}_{\nu \nu} a_{\nu}. \quad (4)
\]

Since \( \hat{A} \) is a triangle, it has a unique two-sided inverse, which we shall denote by \( \hat{A}' \). Thus we may solve (4) for \( a_n \) to obtain

\[
a_n = \sum_{\nu=0}^{n} \hat{a}_{\nu \nu}' X_{\nu}. \quad (5)
\]

Substituting (5) into (3) yields

\[
Y_n = \sum_{\nu=0}^{n} \hat{b}_{\nu \nu} \lambda_{\nu} \hat{a}_{\nu \nu}' = \sum_{\nu=0}^{n} \hat{b}_{\nu \nu} \lambda_{\nu} \left( \sum_{i=0}^{\nu-2} \hat{a}_{\nu i}' X_i + \hat{a}_{\nu \nu-1}' X_{\nu-1} + \hat{a}_{\nu \nu}' X_{\nu} \right)
\]

\[
= \sum_{\nu=0}^{n} \hat{b}_{\nu \nu} \lambda_{\nu} \hat{a}_{\nu \nu}' X_{\nu} + \sum_{\nu=1}^{n} \hat{b}_{\nu \nu} \lambda_{\nu} \hat{a}_{\nu \nu-1}' X_{\nu-1}
\]
\[
\begin{align*}
&+ \sum_{\nu=2}^{n} b_{n\nu} \lambda_{\nu} \sum_{i=0}^{\nu-2} \hat{a}_{\nu i} X_i \\
&= \hat{b}_{nn} \lambda_{nn} \hat{a}_{nn} X_n + \sum_{\nu=0}^{n-1} \hat{b}_{n\nu} \lambda_{\nu} \hat{a}_{\nu \nu} X_{\nu} + \sum_{\nu=0}^{n-1} \hat{b}_{n,\nu+1} \lambda_{\nu+1} \hat{a}_{\nu+1,\nu} X_{\nu} \\
&\quad + \sum_{\nu=2}^{n} b_{n\nu} \lambda_{\nu} \sum_{i=0}^{\nu-2} \hat{a}_{\nu i} X_i \\
&= \frac{b_{nn}}{a_{nn}} \lambda_{nn} X_n + \sum_{\nu=0}^{n-1} \left( \hat{b}_{n\nu} \lambda_{\nu} \hat{a}_{\nu \nu} + \hat{b}_{n,\nu+1} \lambda_{\nu+1} \hat{a}_{\nu+1,\nu} + \hat{b}_{n,\nu} \lambda_{\nu} \hat{a}_{\nu \nu} \right) X_{\nu} \\
&\quad + \sum_{\nu=2}^{n} b_{n\nu} \lambda_{\nu} \sum_{i=0}^{\nu-2} \hat{a}_{\nu i} X_i \\
&= \frac{b_{nn}}{a_{nn}} \lambda_{nn} X_n + \sum_{\nu=0}^{n-1} \frac{\Delta_{\nu} \left( \hat{b}_{n\nu} \lambda_{\nu} \right)}{a_{\nu \nu}} X_{\nu} \\
&\quad + \sum_{\nu=0}^{n-1} \hat{b}_{n,\nu+1} \lambda_{\nu+1} \left( \hat{a}_{\nu \nu} + \hat{a}_{\nu+1,\nu} \right) X_{\nu} + \sum_{\nu=2}^{n} b_{n\nu} \lambda_{\nu} \sum_{i=0}^{\nu-2} \hat{a}_{\nu i} X_i. 
\end{align*}
\]

Using the fact that
\[
a_{\nu \nu} + \hat{a}_{\nu+1,\nu} = \frac{1}{a_{\nu \nu}} \left( \frac{a_{\nu \nu} - a_{\nu+1,\nu}}{a_{\nu+1,\nu+1}} \right),
\]
and substituting (7) into (6), we have
\[
Y_n = \frac{b_{nn}}{a_{nn}} \lambda_{nn} X_n + \sum_{\nu=0}^{n-1} \frac{\Delta_{\nu} \left( \hat{b}_{n\nu} \lambda_{\nu} \right)}{a_{\nu \nu}} X_{\nu} + \sum_{\nu=0}^{n-1} \hat{b}_{n,\nu+1} \lambda_{\nu+1} \left( \frac{a_{\nu \nu} - a_{\nu+1,\nu}}{a_{\nu \nu} a_{\nu+1,\nu+1}} \right) X_{\nu} \\
\quad + \sum_{\nu=2}^{n} b_{n\nu} \lambda_{\nu} \sum_{i=0}^{\nu-2} \hat{a}_{\nu i} X_i \\
= T_{n1} + T_{n2} + T_{n3} + T_{n4}.
\]
By Minkowski’s inequality it is sufficient to show that

\[ \sum_{n=1}^{\infty} n^{k-1} |T_{ni}|^{k} < \infty, \quad i = 1, 2, 3, 4. \]

Using (i)

\[ \sum_{n=1}^{\infty} n^{k-1} |T_{n1}|^{k} = \sum_{n=1}^{\infty} n^{k-1} \left| \frac{b_{nn}}{a_{nn}} \lambda_{n} X_{n} \right|^{k} \]

\[ = O(1) \sum_{n=1}^{\infty} n^{k-1} |X_{n}|^{k} = O(1), \]

since \( \sum a_{n} \) is summable \( |A|_{k} \).

Using (i), (iii), (iv) and H"{o}lder’s inequality,

\[ \sum_{n=1}^{\infty} n^{k-1} |T_{n2}|^{k} = \sum_{n=1}^{\infty} n^{k-1} \left| \sum_{\nu=0}^{n-1} \Delta_{\nu}(\hat{b}_{n\nu} \lambda_{\nu}) \frac{X_{\nu}}{a_{\nu\nu}} \right|^{k} \]

\[ \leq \sum_{n=1}^{\infty} n^{k-1} \left\{ \sum_{\nu=0}^{n-1} (|a_{\nu\nu}|^{-1} |\Delta_{\nu}(\hat{b}_{n\nu} \lambda_{\nu})| |X_{\nu}|)^{k} \right\} \]

\[ = O(1) \sum_{n=1}^{\infty} n^{k-1} \left( \sum_{\nu=0}^{n-1} |b_{\nu\nu} \lambda_{\nu}|^{-k} |\Delta_{\nu}(\hat{b}_{n\nu} \lambda_{\nu})| |X_{\nu}|^{k} \right) \times \]

\[ \times \left( \sum_{\nu=0}^{n-1} |\Delta_{\nu}(\hat{b}_{n\nu} \lambda_{\nu})| \right)^{k-1} \]

\[ = O(1) \sum_{n=1}^{\infty} (n|b_{nn} \lambda_{n}|)^{k-1} \sum_{\nu=0}^{n-1} |b_{\nu\nu} \lambda_{\nu}|^{-k} |\Delta_{\nu}(\hat{b}_{n\nu} \lambda_{\nu})| |X_{\nu}|^{k} \]

\[ = O(1) \sum_{\nu=1}^{\infty} |b_{\nu\nu} \lambda_{\nu}|^{-k} |X_{\nu}|^{k} \sum_{n=\nu+1}^{\infty} (n|b_{nn} \lambda_{n}|)^{k-1} |\Delta_{\nu}(\hat{b}_{n\nu} \lambda_{\nu})| \]

\[ = O(1) \sum_{\nu=1}^{\infty} |b_{\nu\nu} \lambda_{\nu}|^{-k} |X_{\nu}|^{k} \sum_{\nu=1}^{\infty} \nu^{k-1} |b_{\nu\nu} \lambda_{\nu}|^{k} \]

\[ = O(1) \sum_{\nu=1}^{\infty} \nu^{k-1} |X_{\nu}|^{k} = O(1). \]
Using (ii), (v), (vi), (vii) and Hölder’s inequality,
\[
\sum_{n=1}^{\infty} n^{k-1} |T_{n\lambda}|^k = \sum_{n=1}^{\infty} n^{k-1} \left| \sum_{\nu=0}^{n-1} \hat{b}_{n,\nu+1}\lambda_{\nu+1} \left( \frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}} \right) X_{\nu} \right|^k \\
\leq \sum_{n=1}^{\infty} n^{k-1} \left( \sum_{\nu=0}^{n-1} |\hat{b}_{n,\nu+1}\lambda_{\nu+1}| \left| \frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}} \right| |X_{\nu}| \right)^k \\
= O(1) \sum_{n=1}^{\infty} n^{k-1} \left( \sum_{\nu=0}^{n-1} |\hat{b}_{n,\nu+1}\lambda_{\nu+1}| |X_{\nu}| \right)^k \\
= O(1) \sum_{n=1}^{\infty} n^{k-1} \left( \sum_{\nu=0}^{n-1} |b_{\nu\nu}| |\hat{b}_{n,\nu+1}\lambda_{\nu+1}| |X_{\nu}| \right)^k \\
= O(1) \sum_{n=1}^{\infty} n^{k-1} \left( \sum_{\nu=0}^{n-1} |b_{\nu\nu}| \right)^{1-k} \left( \sum_{\nu=0}^{n-1} |\hat{b}_{n,\nu+1}\lambda_{\nu+1}| |X_{\nu}| \right)^k \times \\
\times \left( \sum_{\nu=0}^{n-1} |b_{\nu\nu}| \right)^{k-1} \\
= O(1) \sum_{n=1}^{\infty} \left( n |b_{0n}\lambda_{n+1}| \right)^{k-1} \sum_{\nu=0}^{n-1} |b_{\nu\nu}|^{1-k} |\hat{b}_{n,\nu+1}| |X_{\nu}| \lambda_{\nu+1}|^{k-1} \\
= O(1) \sum_{\nu=0}^{\infty} |b_{\nu\nu}|^{1-k} |\lambda_{\nu+1}| |X_{\nu}| \sum_{n=\nu+1}^{\infty} \left( n |b_{0n}\lambda_{n+1}| \right)^{k-1} |\hat{b}_{n,\nu+1}| \\
= O(1) \sum_{\nu=0}^{\infty} |b_{\nu\nu}|^{1-k} |\lambda_{\nu+1}| |X_{\nu}| \nu^{k-1} |\hat{b}_{\nu+1}\lambda_{\nu+1}|^{k-1} \\
= O(1) \nu^{k-1} |\lambda_{\nu+1}X_{\nu}|^k = O(1).
\]

From (viii),
\[
\sum_{n=1}^{\infty} n^{k-1} |T_{n\lambda}|^k = \sum_{n=1}^{\infty} n^{k-1} \left| \sum_{\nu=0}^{n-2} b_{\nu\nu}\lambda_{\nu} \sum_{i=0}^{\nu-2} \hat{a}_{\nu i} X_i \right|^k = O(1).
\]

\[\Box\]

A weighted mean matrix is a lower triangular matrix with entries \(p_k/P_n\), \(0 \leq k \leq n\), where \(P_n := \sum_{k=0}^{n} p_k\).

**Corollary 1.** Let \(\lambda_n\) be a sequence of constants, \(\{p_n\}\) a sequence of positive constants, \(B\) a triangle satisfying
(i) \( P_n |bn| = O(p_n/|\lambda_n|) \),

\[
\sum_{n=0}^{n-1} |\Delta_\nu (\lambda_\nu \hat{b}_\nu)| = O(|b_{nn} \lambda_n|),
\]

(ii) \( \sum_{n=\nu+1}^n (n|b_{nn} \lambda_n|)^{k-1} |\Delta_\nu (\lambda_\nu \hat{b}_\nu)| = O(\nu^{k-1} |\lambda_\nu b_{\nu\nu}|^k) \),

(iii) \( \sum_{n=\nu+1}^n |b_{\nu\nu} \hat{b}_{n,\nu+1} \lambda_{\nu+1}| = O(|b_{nn} \lambda_n|) \),

(iv) \( \sum_{n=\nu+1}^\infty (n|b_{nn} \lambda_n+1|)^{k-1} |\hat{b}_{n,\nu+1}| = O((\nu|b_{\nu\nu} \lambda_{\nu+1}|)^{k-1}) \),

(v) \( \sum_{\nu=1}^\infty \nu^{k-1} |\lambda_{\nu+1} X_\nu| = O(1) \).

Then \( \lambda \in (|\mathcal{N}|, p_n|k|, |B|, k) \).

**Proof.** Conditions (i), (iii) - (vii) of Theorem 1 reduce to conditions (i) - (vi), respectively of Corollary 1.

With \( A = (\mathcal{N}, p_n) \),

\[
a_{nn} - a_{n+1,n} = \frac{p_n}{P_n} - \frac{p_n}{P_{n+1}} = \frac{p_{nn+1}}{P_nP_{n+1}} = a_{nn+1,n+1},
\]

and condition (ii) of Theorem 1 is automatically satisfied.

A matrix \( A \) is said to be factorable if \( a_{nk} = b_n c_k \) for each \( n \) and \( k \).

Since \( A \) is a weighted mean matrix, \( A \) is a factorable triangle and, as has been shown in [4], its inverse is bidiagonal. Therefore condition (viii) of Theorem 1 is trivially satisfied. \( \square \)

**Corollary 2.** Let \( \lambda_n \) be a sequence of constants, \( \{p_n\} \) a sequence of positive constants, \( A \) a triangle satisfying

(i) \( p_n/(P_n|a_{nn}|) = O(1/|\lambda_n|) \),

(ii) \(|a_{nn} - a_{n+1,n}| = O(|a_{nn} a_{n+1,n+1}|) \),

(iii) \( \sum_{\nu=0}^{n-1} |\Delta_\nu (\lambda_\nu P_{\nu-1})| = O(P_{n-1}|\lambda_n|) \),

(iv) \( |\Delta_\nu (P_{\nu-1} \lambda_\nu)| \sum_{n=\nu+1}^\infty \left( \frac{n p_n |\lambda_n|}{P_n} \right)^{k-1} = O(\nu^{k-1} \left( \frac{P_{\nu} |\lambda_\nu|}{P_{\nu}} \right)^k) \).
(v) \[ \sum_{\nu=0}^{n-1} p_\nu |\lambda_{\nu+1}| = O(P_{n-1} \lambda_{n+1}), \]
(vi) \[ \sum_{\nu=1}^{\infty} n^{k-1} \left( \frac{np_\nu \lambda_{\nu+1}}{P_n} \right)^{k-1} \frac{p_\nu}{P_n P_{n-1}} = O \left( \frac{(\nu p_\nu |\lambda_{\nu+1}|)^{k-1}}{P_n^{k-1}} \right), \]
(vii) \[ \sum_{\nu=1}^{\infty} n^{k-1} |\lambda_{\nu+1} X_{\nu}|^k = O(1), \]
(viii) \[ \sum_{\nu=1}^{\infty} n^{k-1} \left( \frac{p_\nu}{P_n P_{n-1}} \right)^k \left| \sum_{\nu=2}^{n} \lambda_\nu P_{\nu-1} \sum_{i=0}^{\nu-2} \hat{a}_{\nu i} X_i \right|^k = O(1). \]

Then \( \lambda \in (|A|, |\mathcal{N}, P_n|, k) \).

**Proof.** With \( B = (\mathcal{N}, P_n) \), conditions (i) - (viii) of Theorem 1 reduce to conditions (i) - (viii), respectively of Corollary 2, since

\[ \hat{b}_{\nu \nu} = \frac{p_\nu P_{\nu-1}}{P_n P_{n-1}}. \]

\[ \square \]

**Corollary 3.** Let \( q_n = 1 \) for each \( n \), \( \{p_n\} \) a positive sequence satisfying conditions (iii)-(vi) of Corollary 2,

(i) \[ \frac{np_n |\lambda_n|}{P_n} = O(1), \]
(ii) \[ \sum_{\nu=1}^{\infty} n^{k-1} |\lambda_\nu X_{\nu}|^k = O(1). \]

Then \( \lambda \in (|C, 1|, |\mathcal{N}, P_n|, k) \).

**Proof.** With \( A = (C, 1) \), condition (i) of Corollary 2 becomes condition (i) of Corollary 3.

Note that

\[ a_{nn} - a_{n+1,n} = \frac{p_n}{P_n} - \frac{p_n}{P_{n+1}} = \frac{p_n P_{n+1}}{P_n P_{n+1}} = a_{nn} a_{n+1,n+1}, \]

and condition (ii) of Corollary 2 is automatically satisfied.

Since the inverse of \((C, 1)\) is bidiagonal, condition (vii) of Corollary 2 is automatically satisfied. \[ \square \]

**Corollary 4.** Let \( \{p_n\} \) be a positive sequence, \( q_n = 1 \) for each \( n \), satisfying
(i) \( \frac{P_n|\lambda_n|}{np_n} = O(1) \),

(ii) \( \sum_{\nu=0}^{n-1} |\Delta_{\nu}(\nu\lambda_{\nu})| = O(n|\lambda_n|) \),

(iii) \( |\Delta_{\nu}(\nu\lambda_{\nu})| \sum_{n=\nu+1}^{\infty} \frac{|\lambda_n|^{k-1}}{n(n+1)} = O\left( \frac{|\lambda_{\nu}|^k}{\nu} \right) \),

(iv) \( \sum_{\nu=0}^{n-1} |\lambda_{\nu+1}| = O(n|\lambda_{n+1}|) \),

(v) \( \sum_{\nu=0}^{n} \frac{|\lambda_{n+1}|^k}{n(n+1)^k} = O\left( \left( \frac{|\lambda_{\nu+1}|}{\nu} \right)^{k-1} \right) \),

(vi) \( \sum_{\nu=1}^{\infty} \nu^{k-1}|\lambda_{\nu+1}X_{\nu}|^k = O(1) \).

Then \( \lambda \in \overline{\{N, p_n|k, |C, 1|}\} \).

With \( B = (C, 1) \), the conditions of Corollary 1 reduce to those of Corollary 4.

We now turn our attention to obtaining necessary conditions.

Theorem 2. Let \( A \) and \( B \) be two lower triangular matrices with \( A \) satisfying

\[
\sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_{\nu}\hat{a}_{n\nu}|^k = O(|a_{\nu\nu}|^k).
\]

Then necessary conditions for \( \lambda \in (A|k, |B|) \) are

(i) \( |b_{\nu\nu}\lambda_{\nu}| = O(|a_{\nu\nu}|) \),

(ii) \( \left( \sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_{\nu}\hat{b}_{n\nu}\lambda_{\nu}|^{k} \right)^{1/k} = O(|a_{\nu\nu}|^{1-1/k}) \),

(iii) \( \sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k = O\left( \sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{a}_{n,\nu+1}|^k \right) \).

Proof. For \( k \geq 1 \) define

\[
A^* = \left\{ \{a_i\} : \sum a_i \text{ is summable} |A|_k \right\},
\]

\[
B^* = \left\{ \{b_i\} : \sum b_i\lambda_i \text{ is summable} |B|_k \right\}.
\]
With \( Y_n \) and \( X_n \) as defined by (3) and (4), the spaces \( A^* \) and \( B^* \) are BK-spaces, with norms given by
\[
\|a\|_1 = \left\{ |X_0|^k + \sum_{n=1}^{\infty} n^{k-1}|X_n|^k \right\}^{1/k}, \tag{9}
\]
and
\[
\|a\|_2 = \left\{ |Y_0|^k + \sum_{n=1}^{\infty} n^{k-1}|Y_n|^k \right\}^{1/k}, \tag{10}
\]
respectively.

From the hypothesis of the theorem, \( \|a\|_1 < \infty \) implies that \( \|a\|_2 < \infty \).

The inclusion map \( i : A^* \to B^* \) defined by \( i(x) = x \) is continuous, since \( A^* \) and \( B^* \) are BK-spaces. Applying the closed graph theorem, there exists a constant \( K > 0 \) such that
\[
\|a\|_2 \leq K \|a\|_1. \tag{11}
\]

Let \( e_n \) denote the \( n \)-th coordinate vector. From (3) and (4), with \( \{a_n\} \) defined by \( a_n = e_n - e_{n+1}, n = \nu \), \( a_n = 0 \) otherwise, we have

\[
X_n = \begin{cases} 
0, & n < \nu, \\
\hat{a}_{\nu\nu}, & n = \nu, \\
\Delta_\nu \hat{a}_{\nu\nu}, & n > \nu,
\end{cases}
\]
and

\[
Y_n = \begin{cases} 
0, & n < \nu, \\
\hat{b}_{\nu\nu}, & n = \nu, \\
\Delta_\nu \hat{b}_{\nu\nu}, & n > \nu.
\end{cases}
\]

From (9) and (10),
\[
\|a\|_1 = \nu^{k-1}|a_{\nu\nu}|^k + \sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu \hat{a}_{\nu\nu}|^k \right\}^{1/k},
\]
and
\[
\|a\|_2 = \nu^{k-1}|b_{\nu\nu}|^k + \sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu \hat{b}_{\nu\nu}|^k \right\}^{1/k},
\]
recalling that \( \hat{b}_{\nu\nu} = \tilde{b}_{\nu\nu} = \overline{b}_{\nu\nu} \).

From (11), using (8), we obtain
\[
\nu^{k-1}|\nu_{\nu\nu}|^k + \sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu \overline{b}_{\nu\nu}|^k.
\]
\[ \leq K^k (\nu^{k-1} |a_{\nu\nu}|^k + \sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu \hat{a}_{\nu\nu}|^k) \]
\[ \leq K^k (\nu^{k-1} |a_{\nu\nu}|^k + O(1) |a_{\nu\nu}|^k) \]
\[ = O(|a_{\nu\nu}|^k (\nu^{k-1} + 1)) \]
\[ = O(\nu^{k-1} |a_{\nu\nu}|^k). \]

The above inequality will be true if and only if each term on the left hand side is \( O(\nu^{k-1} |a_{\nu\nu}|^k) \). Using the first term,
\[ \nu^{k-1} |b_{\nu\nu} \lambda_{\nu}|^k = O(\nu^{k-1} |a_{\nu\nu}|^k), \]
which implies that \( |b_{\nu\nu} \lambda_{\nu}| = O(|a_{\nu\nu}|) \), and (i) is necessary.

Using the second term we obtain
\[ \left( \sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu (\hat{b}_{\nu\nu} \lambda_{\nu})|^k \right)^{1/k} = O(\nu^{1-1/k} |a_{\nu\nu}|), \]
which is condition (ii).

If we now define \( a_n = e_{n+1} \) for \( n = \nu \), \( a_n = 0 \) otherwise, then, from (3) and (4) we obtain
\[ X_n = \begin{cases} 0, & n \leq \nu, \\ \hat{a}_{n,\nu+1}, & n > \nu, \end{cases} \]
and
\[ Y_n = \begin{cases} 0, & n \leq \nu, \\ \hat{b}_{n,\nu+1} \lambda_{\nu+1}, & n > \nu. \end{cases} \]

The corresponding norms are
\[ \|a\|_1 = \left\{ \sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{a}_{n,\nu+1}|^k \right\}^{1/k}, \]
and
\[ \|a\|_2 = \left\{ \sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{b}_{n,\nu+1} \lambda_{\nu+1}|^k \right\}^{1/k}. \]

Applying (11),
\[ \left\{ \sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{b}_{n,\nu+1} \lambda_{\nu+1}|^k \right\}^{1/k} \leq K \left\{ \sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{a}_{n,\nu+1}|^k \right\}^{1/k}, \]
which implies condition (iii). \( \square \)
Corollary 5. Let $B$ be a lower triangular matrix, $\{p_n\}$ a sequence satisfying
\[\sum_{n=\nu+1}^{\infty} n^{k-1} \left( \frac{p_n}{P_n P_{n-1}} \right)^k = O \left( \frac{1}{P_k} \right). \tag{12}\]

Then necessary conditions for $\lambda \in (|N|, p_n|, |B|_k)$ are
\[(i) \quad |b_{\nu\nu}\lambda_\nu| = O \left( \frac{P_\nu}{P_\nu} \right),\]
\[(ii) \quad \left( \sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu (b_{n\nu}\lambda_\nu)|^k \right)^{1/k} = O \left( \frac{P_\nu}{P_\nu} \right),\]
\[(iii) \quad \sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k = O(1).\]

Proof. With $A = (N, p_n)$, equation (8) becomes (12), and conditions (i)-(iii) of Theorem 2 become conditions (i)-(iii) of Corollary 10, respectively. \qed

Corollary 6. Let $1 \leq k < \infty$, $\{p_n\}$ a positive sequence. Then $\lambda \in (|N|, p_n|, |B|_k)$ if and only if
\[(i) \quad |b_{\nu\nu}\lambda_\nu| = O \left( \frac{P_\nu}{P_\nu} \right),\]
\[(ii) \quad \left( \sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu (b_{n\nu}\lambda_\nu)|^k \right)^{1/k} = O \left( \frac{P_\nu}{P_\nu} \right),\]
\[(iii) \quad \sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k = O(1).\]

Every summability factor theorem becomes an inclusion theorem by setting each $\lambda_n = 1$.

Corollary 7. Let $A$ and $B$ be triangles satisfying
\[
(i) \quad \frac{|a_{nn}|}{|b_{nn}|} = O(1),
(ii) \quad \frac{b_{n+1,n} - b_{nn}}{b_{nn} b_{n+1,n+1}} = O(1),
(iii) \quad \sum_{\nu=0}^{n-1} |\Delta_\nu \hat{a}_{n\nu}| = O(|a_{nn}|),
(iv) \quad \sum_{n=\nu+1}^{\infty} (n|a_{nn}|)^{k-1} |\Delta_\nu \hat{a}_{n\nu}| = O\left( \nu^{k-1} |a_{\nu\nu}|^k \right),
\]
\((v)\) \(\sum_{\nu=0}^{n} |a_{\nu\nu}| \tilde{a}_{n,\nu+1} = O(|a_{nn}|),\)
\[(vi)\] \(\sum_{n=\nu+1}^{\infty} (n|a_{nn}|)^{k-1} |\tilde{a}_{n,\nu+1}| = O((\nu|a_{\nu\nu}|)^{k-1}),\)
\[(vii)\] \(\sum_{n=1}^{\infty} n^{k-1} \left| \sum_{\nu=2}^{n-2} \sum_{r=1}^{\nu \nu} \tilde{b}_{\nu r} X_r \right|^k = O(1).\)

Then \(\sum a_n\) summable \(|B|k\) implies that it is summable \(|A|k, k \geq 1.\)

Corollary 7 is Theorem 1 of [3].

**Corollary 8.** Let \(\{p_n\}\) be a positive sequence, \(T\) a nonnegative triangle satisfying

\[(i)\] \(t_{ni} \geq t_{n+1,i}, \quad n \geq i, i = 0, 1, \ldots,\)
\[(ii)\] \(P_n t_{nn} = O(p_n),\)
\[(iii)\] \(\tilde{t}_{n0} = \tilde{t}_{n-1,0}, \quad n = 1, 2, \ldots,\)
\[(iv)\] \(\sum_{\nu=1}^{n-1} t_{\nu\nu} |\tilde{t}_{n,\nu}| = O(t_{nn}),\)
\[(v)\] \(\sum_{n=\nu+1}^{\infty} (mt_{nn})^{k-1} |\Delta_\nu \tilde{t}_{n\nu}| = O(\nu^{k-1} t_{\nu\nu}^k),\)
\[(vi)\] \(\sum_{n=\nu+1}^{\infty} (mt_{nn})^{k-1} |\tilde{t}_{n,\nu}| = O((\nu t_{\nu\nu})^{k-1}).\)

Then \(\sum a_n\) summable \(|\mathcal{N}, p_n|k\) implies \(\sum a_n\) is summable \(|T|k, k \geq 1.\)

**Proof.** Since each \(\lambda_n = 1,\) condition (vi) of Corollary 1 simply states that \(\sum a_n\) is summable \(|\mathcal{N}, p_n|k.\)

Condition (i) of Corollary 1 reduces to condition (ii) of Corollary 6.

Note that
\[
\Delta_\nu \tilde{t}_{n\nu} = \tilde{t}_{n\nu} - \tilde{t}_{n,\nu+1} = \tilde{t}_{n\nu} - \tilde{t}_{n-1,\nu} - \tilde{t}_{n,\nu+1} + \tilde{t}_{n-1,\nu+1} \\
= \sum_{i=\nu}^{n} t_{ni} - \sum_{i=\nu+1}^{n-1} t_{n-1,i} - \sum_{i=\nu}^{n} t_{ni} + \sum_{i=\nu+1}^{n-1} t_{n-1,i} \\
= t_{n\nu} - t_{n-1,\nu} \geq 0.
\]
Therefore, from (i) and (iii) of Corollary 6,
\[ \sum_{\nu=0}^{n-1} |\Delta_{\nu} \hat{t}_{n\nu}| = \sum_{\nu=0}^{n-1} |t_{n\nu} - t_{n-1,\nu}| = \sum_{\nu=0}^{n-1} t_{n-1,\nu} - \sum_{\nu=0}^{n-1} t_{n\nu} = \hat{t}_{n-1,0} - \hat{t}_{n0} + t_{nn} = t_{nn}, \]
and condition (ii) of Corollary 1 is satisfied.

Condition (iii) of Corollary 1 reduces to condition (v) of Corollary 6.

Using condition (ii) of Corollary 1, condition (iv) of Corollary 6, and the fact that condition (iii) of Corollary 6 implies that \( \hat{t}_{n0} = 0 \),
\[ \sum_{\nu=0}^{n-1} t_{\nu\nu} \hat{t}_{n,\nu+1} = \sum_{\nu=0}^{n-1} t_{\nu\nu} (\hat{t}_{n,\nu+1} - \hat{t}_{n\nu}) + \sum_{\nu=0}^{n-1} t_{\nu\nu} \hat{t}_{n\nu} = O(t_{nn}), \]
and condition (iv) of Corollary 1 is satisfied.

Using condition (iv) of Corollary 1 and condition (v) of Corollary 6,
\[ \sum_{n=\nu+1}^{\infty} (nt_{nn})^{k-1} \hat{t}_{n,\nu+1} = \sum_{n=\nu+1}^{\infty} (nt_{nn})^{k-1} |\Delta_{\nu} \hat{t}_{n\nu}| + \sum_{n=\nu+1}^{\infty} (nt_{nn})^{k-1} \hat{t}_{n\nu} = O((\nu t_{nn})^{k-1}), \]
and condition (v) of Corollary 1 is satisfied.

\[ \square \]

**Remark 1.** Corollary 6 is equivalent to the corrected version of the Theorem in [1], which appears in [2].

**Corollary 9.** Let \( A \) and \( B \) be two lower triangular matrices, \( A \) satisfying (8). Necessary conditions for \( \sum a_n \) summable \( |A|_k \) to imply that \( \sum a_n \) is summable \( |B|_k \) are

(i) \( |b_{\nu\nu}| = O(|a_{\nu\nu}|) \),

(ii) \( \sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_{\nu} \hat{b}_{n\nu}|^k = O(|a_{\nu\nu}|^{k2^{k-1}}) \),

(iii) \( \sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{b}_{n,\nu+1}|^k = O\left( \sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{a}_{n,\nu+1}|^k \right) \).

To prove the corollary simply put \( \lambda_n = 1 \) in Theorem 2.

**Corollary 10.** Let \( B \) be a lower triangular matrix, \( A \) a weighted mean matrix with \( \{p_n\} \) a sequence satisfying (8). Then necessary conditions for \( \sum a_n \) summable \( |N, p_n|_k \) to imply that \( \sum a_n \) is summable \( |B|_k \) are
(i) \[ \frac{P_\nu|b_{\nu\nu}|}{P_\nu} = O(1), \]

(ii) \[ \sum_{n=\nu+1}^\infty n^{k-1}|\Delta_\nu \hat{b}_{n\nu}|^k = O\left(\nu^{k-1}\left(\frac{P_\nu}{P_\nu}\right)^k\right), \]

(iii) \[ \sum_{n=\nu+1}^\infty n^{k-1}|\Delta_\nu \hat{b}_{n,\nu+1}|^k = O(1). \]

To prove the corollary set \( \lambda_n = 1 \) in Corollary 5.

REFERENCES


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